Colophon

This book was constructed using XeLaTeX and output directly to PDF. Latin script was set in Calisto. Some mathematical graphics were created with Mathematica and exported to Encapsulated PostScript.

My worked solutions largely follow the Answers to Problems in the textbook. The Exercises were often of a fair level of complexity and the solutions often sufficiently ingenious, calling upon enough foundational mathematics that had faded from memory, that I found it useful to . I learned as much, if not more, from this process than if I'd simply read the text. These worked solutions represent my best effort at interpreting the author's answers, occasionally expounding on them, and even more rarely correcting errors. Do not expect them to be error-free. I welcome corrections or suggestions.

Answers to Exercises

§8 Examples of Special Curves

☐ 8.1a Let:

$$\mathbf{x}t = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}, 0\right) \tag{1}$$

$$x = \frac{3t}{1+t^3} \Rightarrow (1+t^3)x = 3t \Rightarrow xt^3 - 3t + x = 0 \tag{2}$$

$$y = \frac{3t^2}{1+t^3} \Rightarrow (1+t^3)y = 3t^2 \Rightarrow yt^3 - 3t^2 + y = 0 \tag{3}$$

$$\begin{vmatrix} x & 0 & -3 & x & & \\ & x & 0 & -3 & x & \\ & & x & 0 & -3 & x \\ y & -3 & 0 & y & & \\ & y & -3 & 0 & y & \\ & & y & -3 & 0 & y \end{vmatrix} = x\begin{vmatrix} x & 0 & -3 & x & \\ & x & 0 & -3 & x \\ -3 & 0 & y & & \\ y & -3 & 0 & y & \\ & y & -3 & 0 & y \end{vmatrix} - y\begin{vmatrix} 0 & -3 & x & & \\ x & 0 & -3 & x & \\ & x & 0 & -3 & x \\ y & -3 & 0 & y & \\ & y & -3 & 0 & y \end{vmatrix} \tag{4}$$

$$= -x^2\begin{vmatrix} x & 0 & -3 & x \\ -3 & 0 & y & \\ y & -3 & 0 & y \\ & y & -3 & 0 \end{vmatrix} + xy\begin{vmatrix} x & 0 & -3 & x \\ & x & 0 & -3 \\ -3 & 0 & y & \\ y & -3 & 0 & y \end{vmatrix} \tag{5}$$

$$- xy\begin{vmatrix} 0 & -3 & x & \\ x & 0 & -3 & x \\ y & -3 & 0 & y \\ & y & -3 & 0 \end{vmatrix} - y^2\begin{vmatrix} 0 & -3 & x & \\ x & 0 & -3 & x \\ & x & 0 & -3 \\ y & -3 & 0 & y \end{vmatrix} \tag{6}$$

$$= x^3\begin{vmatrix} -3 & 0 & y \\ y & -3 & 0 \\ & y & -3 \end{vmatrix} + x^2y\begin{vmatrix} x & 0 & -3 \\ -3 & 0 & y \\ & y & -3 \end{vmatrix} + x^2y\begin{vmatrix} x & -3 & x \\ -3 & y & \\ y & 0 & y \end{vmatrix} - 3xy\begin{vmatrix} x & -3 & x \\ & 0 & -3 \\ -3 & y & \end{vmatrix} \tag{7}$$

$$+x^2y\begin{vmatrix} -3 & x & \\ -3 & 0 & y \\ y & -3 & 0 \end{vmatrix} - xy^2\begin{vmatrix} -3 & x & \\ 0 & -3 & x \\ y & -3 & 0 \end{vmatrix} + xy^2\begin{vmatrix} -3 & x & \\ x & 0 & -3 \\ -3 & 0 & y \end{vmatrix} + y^3\begin{vmatrix} -3 & x & \\ 0 & -3 & x \\ x & 0 & -3 \end{vmatrix} \tag{8}$$

$$
\begin{aligned}
&= x^3\left(-3\begin{vmatrix}-3 & 0\\ y & -3\end{vmatrix} - y\begin{vmatrix}0 & y\\ y & -3\end{vmatrix}\right) - x^2y^2\begin{vmatrix}x & -3\\ -3 & y\end{vmatrix} + x^2y\left(x\begin{vmatrix}-3 & y\\ y & 0\end{vmatrix} + y\begin{vmatrix}x & -3\\ -3 & y\end{vmatrix}\right) && (9)\\
&- 9xy\begin{vmatrix}x & -3\\ -3 & y\end{vmatrix} - x^2y^2\begin{vmatrix}-3 & x\\ y & -3\end{vmatrix} + x^2y^2\begin{vmatrix}-3 & x\\ y & -3\end{vmatrix} - x^2y^2\begin{vmatrix}x & -3\\ -3 & y\end{vmatrix} && (10)\\
&+ y^3\left(-3\begin{vmatrix}-3 & x\\ 0 & -3\end{vmatrix} - x\begin{vmatrix}0 & x\\ x & -3\end{vmatrix}\right) && (11)\\
& && (12)
\end{aligned}
$$

$$
\begin{aligned}
&= -3x^3\begin{vmatrix}-3 & 0\\ y & -3\end{vmatrix} - x^3y\begin{vmatrix}0 & y\\ y & -3\end{vmatrix} - 2x^2y^2\begin{vmatrix}x & -3\\ -3 & y\end{vmatrix} + x^3y\begin{vmatrix}-3 & y\\ y & 0\end{vmatrix} && (13)\\
&+ x^2y^2\begin{vmatrix}x & -3\\ -3 & y\end{vmatrix} - 9xy\begin{vmatrix}x & -3\\ -3 & y\end{vmatrix} - 3y^3\begin{vmatrix}-3 & x\\ 0 & -3\end{vmatrix} - xy^3\begin{vmatrix}0 & x\\ x & -3\end{vmatrix} && (14)\\
&= -3x^3\begin{vmatrix}-3 & 0\\ y & -3\end{vmatrix} - x^2y^2\begin{vmatrix}x & -3\\ -3 & y\end{vmatrix} - 9xy\begin{vmatrix}x & -3\\ -3 & y\end{vmatrix} - 3y^3\begin{vmatrix}-3 & x\\ 0 & -3\end{vmatrix} - xy^3\begin{vmatrix}0 & x\\ x & -3\end{vmatrix} && (15)\\
&= -3x^3\cdot 9 - x^2y^2(xy-9) - 9xy(xy-9) - 3y^3\cdot 9 - xy^3\cdot -x^2 && (16)\\
&= -27x^3 - x^3y^3 + 9x^2y^2 - 9x^2y^2 + 81xy - 27y^3 + x^3y^3 && (17)\\
&= -27x^3 - 27y^3 + 81xy = 0 && (18)\\
&\Rightarrow x^3 + y^3 - 3xy = 0 and z = 0. && (19)
\end{aligned}
$$

☐ 8.1b Let:

$$
\mathbf{x}t = \left(\frac{at}{t^2+1}, \frac{b(t^2-1)}{t^2+1}, 0\right) a, b \neq 0 \qquad (20)
$$

$$
\begin{aligned}
x &= \frac{at}{t^2+1} \Rightarrow (t^2+1)x = at \Rightarrow xt^2 - at + x = 0 && (21)\\
y &= \frac{b(t^2+1)}{t^2+1} \Rightarrow (t^2+1)y = b(t^2-1) \Rightarrow (y-b)t^2 + (y+b) = 0 && (22)
\end{aligned}
$$

$$
\begin{vmatrix} x & -a & x & \\ & x & -a & x\\ y-b & 0 & y+b & \\ & y-b & 0 & y+b \end{vmatrix} = x\begin{vmatrix} x & -a & x\\ 0 & y+b & \\ y-b & 0 & y+b\end{vmatrix} + (y-b)\begin{vmatrix} -a & x & \\ x & -a & x\\ y-b & 0 & y+b\end{vmatrix} \qquad (23)
$$

$$
\qquad (24)
$$

$$
\begin{aligned}
&= x(y+b)\begin{vmatrix}x & x\\ y-b & y+b\end{vmatrix} - a(y-b)\begin{vmatrix}-a & x\\ 0 & y+b\end{vmatrix} - x(y-b)\begin{vmatrix}x & x\\ y-b & y+b\end{vmatrix} && (25)\\
&= x(y+b)\left(x(y+b) - x(y-b)\right) - a(y-b)\left(-a(y+b)\right) - x(y-b)\left(x(y+b) - x(y-b)\right) && (26)\\
&= x^2(y+b)^2 - x^2(y+b)(y-b) + a^2(y+b)(y-b) - x^2(y+b)(y-b) + x^2(y-b)^2 && (27)\\
&= x^2(y+b)^2 + (a^2 - 2x^2)(y^2 - b^2) + x^2(y-b)^2 && (28)\\
&= x^2(y^2 + 2by + b^2 + y^2 - 2by + b^2) + a^2(y^2 - b^2) - 2x^2(y^2 - b^2) && (29)\\
&= 2x^2y^2 + 2b^2x^2 + a^2y^2 - a^2b^2 - 2x^2y^2 + 2b^2x^2 && (30)\\
&= 4b^2x^2 + a^2y^2 - a^2b^2. && (31)
\end{aligned}
$$

§9 Arc Length

☐ 9.1 Let:

$$\mathbf{x}t = (r\cos t, r\sin t, ct)\ c \neq 0 \tag{32}$$

$$st = \int_{t_0}^{t} \sqrt{\mathbf{x}' \cdot \mathbf{x}'}\, \mathrm{d}t \tag{33}$$

$$\mathbf{x}'t = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(r\cos t, r\sin t, ct) = (-r\sin t, r\cos t, c) \tag{34}$$

$$\begin{aligned} \mathbf{x}' \cdot \mathbf{x}' &= (-r\sin t, r\cos t, c) \cdot (-r\sin t, r\cos t, c) && (35)\\ &= r^2 \sin^2 t + r^2 \cos^2 t + c^2 && (36)\\ &= r^2(\sin^2 t + \cos^2 t) + c^2 = r^2 + c^2 && (37) \end{aligned}$$

$$s = \int_0^t \sqrt{\mathbf{x}' \cdot \mathbf{x}'}\, \mathrm{d}t = \int_0^t \sqrt{r^2 + c^2}\, \mathrm{d}t = t\sqrt{r^2 + c^2} \tag{38}$$

$$t \overset{c\neq 0}{=\!=} \frac{1}{\sqrt{r^2 + c^2}} s = \omega s \left(\omega = \frac{1}{\sqrt{r^2 + c^2}}\right) \tag{39}$$

$$\mathbf{x}s = (r\cos\omega s, r\sin\omega s, \omega cs) \tag{40}$$

☐ 9.2 Let:

$$\mathbf{x} = \left(t, a\cosh\frac{t}{a}, 0\right) \tag{41}$$

$$\mathbf{x}' = \frac{\mathrm{d}}{\mathrm{d}t}\left(t, a\cosh\frac{t}{a}, 0\right) = \left(1, a\sinh\frac{t}{a} \cdot \frac{1}{a}, 0\right) = \left(1, \sinh\frac{t}{a}, 0\right) \tag{42}$$

$$\mathbf{x}' \cdot \mathbf{x}' = 1 + \sinh^2\frac{t}{a} = \cosh^2\frac{t}{a} \tag{43}$$

$$\sqrt{\mathbf{x}' \cdot \mathbf{x}'} = \sqrt{\cosh^2\frac{t}{a}} = \cosh\frac{t}{a} \tag{44}$$

$$st = \int_0^t \sqrt{\mathbf{x}' \cdot \mathbf{x}'}\, \mathrm{d}t = \int_0^t \cosh\frac{t}{a}\, \mathrm{d}t = a\sinh\frac{t}{a} \tag{45}$$

☐ 9.3 Let:

$$\mathbf{x} = (a\cos\phi, b\sin\phi, 0)\text{assume} a \geq b,\ a, b \neq 0 \tag{46}$$

$$\mathbf{x}' = \frac{\mathrm{d}}{\mathrm{d}\phi}(a\cos\phi, b\sin\phi, 0) = (-a\sin\phi, b\cos\phi, 0) \tag{47}$$

$$\begin{aligned} \mathbf{x}' \cdot \mathbf{x}' &= a^2\sin^2\phi + b^2\cos^2\phi && (48)\\ &= a^2\sin^2\phi + a^2\cos^2\phi - a^2\cos^2\phi + b^2\cos^2\phi && (49)\\ &= a^2 - (a^2 - b^2)\cos^2\phi = a^2\left(1 - \frac{a^2 - b^2}{a^2}\cos^2\phi\right) && (50)\\ &= a^2(1 - \epsilon^2\cos^2\phi), \text{where}\epsilon^2 = \frac{a^2 - b^2}{a^2} \Leftarrow \epsilon = \sqrt{\frac{a^2 - b^2}{a^2}} = \frac{1}{a}\sqrt{a^2 - b^2} && (51) \end{aligned}$$

☐ 9.4 Let:

$$\mathbf{x}t = (t, t\sin\frac{1}{t}, 0) \tag{52}$$

Without loss of generality, determine the length of the arc from $t_0 = \frac{2}{\pi}$ to $t = 0$. Break the arc into a sequence of chords:

$$t_i = \frac{2}{(2i+1)\pi} \tag{53}$$

$$l_i = |y(t_{i+1}) - y(t_i)| > 2\,|y(t_{i+1})| = 2t_{i+1} = 2\frac{2}{(2i+1)\pi} = \frac{4}{(2i+1)\pi} \tag{54}$$

$$= \frac{1}{\pi}\frac{4}{2i+1} = \frac{2}{\pi}\frac{1}{i+\frac{1}{2}} > \frac{2}{\pi}\frac{1}{i+1} \tag{55}$$

$$\sum_{i=0}^{\infty} l_i = \sum \frac{2}{\pi}\frac{1}{i+1} = \frac{2}{\pi}\sum \frac{1}{i+1} = \infty \tag{56}$$

so the arc does not have a well-defined length.

§10 Tangent and Normal Plane

△ Let:

$$\mathbf{x}t = (r\cos t, r\sin t, ct) \text{ where } c \neq = \tag{57}$$

$$\mathbf{x}' = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = (-r\sin t, r\cos t, c) \tag{58}$$

$$s'^2 = \mathbf{x}' \cdot \mathbf{x}' = r^2\sin^2 t + r^2\cos^2 t + c^2 = r^2 + c^2 \Rightarrow s' = \sqrt{r^2 + c^2} \tag{59}$$

$$\mathbf{t} = \frac{\mathbf{x}'}{|\mathbf{x}'|} = \frac{\mathbf{x}'}{s'} = \frac{1}{\sqrt{r^2+c^2}}(-r\sin t, r\cos t, c) \tag{60}$$

☐ 10.1 Let: The curve is given by the parametric representation:

$$\mathbf{x}t = (t, t^2, 0) \tag{61}$$

A representation of the tangent is found by:

$$\mathbf{x}'t = \frac{\mathrm{d}\mathbf{x}}{(t)} = (1, 2t, 0) \tag{62}$$

$$\mathbf{y}u = \mathbf{x} + u\mathbf{x}' \Rightarrow \mathbf{y}_{t=1} = \mathbf{x}_{t=1} + u\mathbf{x}'_{t=1} \tag{63}$$

$$= (1, 1, 0) + u(1, 2, 0) = (1+u, 1+2u, 0) \tag{64}$$

At $y = 0$:

$$1 + 2u = 0 \Rightarrow 2u = -1 \Rightarrow u = -\tfrac{1}{2} \tag{65}$$

so the point of intersection is $(\frac{1}{2}, 0, 0)$.

☐ 10.2 The parametric representation of the curve can be written as:

$$\mathbf{x}t = (t, \beta t^2, \gamma t^n) \text{where} \beta, \gamma \in \mathbb{N} \tag{66}$$

A representation of the tangent is found by:

$$\mathbf{x}'t = (1, 2\beta t, n\gamma t^{n-1}) \tag{67}$$

$$\begin{aligned} \mathbf{y}_t u &= \mathbf{x}_t + \mathbf{x}'_t u && (68)\\ &= (t, \beta t^2, \gamma t^n) + u(1, 2\beta t, n\gamma t^{n-1}) && (69)\\ &= (t+u, \beta t^2 + 2u\beta t, \gamma t^n + un\gamma t^{n-1}) && (70) \end{aligned}$$

The intersection with the x, y plane:

$$\begin{aligned} \gamma t^n + un\gamma t^{n-1} = 0 &\Rightarrow t^{n-1}(t+un) = 0 && (71)\\ &\Rightarrow t^{n-1} = 0 \vee t = -un && (72)\\ &\Rightarrow t = 0 \vee t = -un && (73)\\ &\Rightarrow t = 0 \vee u = -\frac{t}{n} && (74) \end{aligned}$$

So

$$\begin{aligned} \mathbf{y}_t(u = -\frac{t}{n}) &= \left(t - \frac{t}{n}, \beta t^2 - \frac{t}{n} \cdot 2\beta t, \gamma t^n - \frac{t}{n} n\gamma t^{n-1}\right) && (75)\\ &= \left(t(1-\frac{1}{n}), \beta t^2 - \frac{2\beta t^2}{n}, \gamma t^n - \gamma t^n\right) && (76)\\ &= \left(\frac{n-1}{n}t, (1-\frac{2}{n})\beta t^2, 0\right) = \left(\frac{n-1}{n}t, \frac{n-2}{n}\beta t^2, 0\right) && (77) \end{aligned}$$

which also implies the case of $\mathbf{x}(t=0) = (0,0,0)$.

☐ 10.3 The two circles can be described parametrically by:

$$\begin{aligned} \mathbf{x}t &= (\cos t, \sin t) && (78)\\ \tilde{\mathbf{x}}t &= (\cos t + 1, \sin t - 1) && (79) \end{aligned}$$

The tangents are parallel to:

$$\begin{aligned} \mathbf{y}'t &= (-\sin t, \cos t) && (80)\\ \tilde{\mathbf{y}}'t &= (-\sin t, \cos t) && (81) \end{aligned}$$

So without loss of generality, find a tangent $\mathbf{y}_t$ to $\mathbf{x}t$ that is also tangent to $\tilde{\mathbf{x}}t^*$. The worked solution considers only the case that $t = t^*$, but this fails in other configurations--- such as the case where the second circle is located at $(-1, -1)$. The tangents can only coincide if they are parallel, i.c. $\tilde{\mathbf{x}}' = \pm\mathbf{x}'$ $\Rightarrow t^* =_\pi t$ and if any point coincides, e.g. $\tilde{\mathbf{x}}t^* \in \{_{u \in \mathbb{R}}\ \mathbf{y}_t\}$. The case $t^* =_{2\pi} \pi + t$ fails on that score, so we consider only the case of $t^* =_{2\pi} t \Leftarrow t^* = t$:

$$\tilde{\mathbf{x}}t \in \{_u\ \mathbf{y}_t u\} \Rightarrow \tilde{\mathbf{x}}t \in \{_u\ \mathbf{x}t + u\mathbf{x}'\} \tag{82}$$

$$\Rightarrow \begin{cases} \cos t + 1 = \cos t - u \sin t \\ \sin t - 1 = \sin t + u \cos t \end{cases} \Rightarrow \begin{cases} u \sin t = -1 \\ u \cos t = -1 \end{cases} \tag{83}$$

$$\Rightarrow \frac{1}{\sin t} = \frac{1}{\cos t} \Rightarrow \sin t = \cos t \text{where} \sin t, \cos t \neq 0 \Rightarrow t =_\pi \tfrac{1}{4}\pi \tag{84}$$

$$\mathbf{y}_{\frac{1}{4}\pi} u = (\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}u\sqrt{2}, \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}u\sqrt{2}) \tag{85}$$

$$\Rightarrow \begin{cases} x = \frac{1}{2}\sqrt{2} - \frac{1}{2}u\sqrt{2} \\ y = \frac{1}{2}\sqrt{2} + \frac{1}{2}u\sqrt{2} \end{cases} \Rightarrow \begin{cases} x\sqrt{2} = 1 - u \\ y\sqrt{2} = 1 + u \end{cases} \Rightarrow \begin{cases} u = 1 - x\sqrt{2} \\ u = -1 + y\sqrt{2} \end{cases} \tag{86}$$

$$\Rightarrow 1 - x\sqrt{2} = -1 + y\sqrt{2} \Rightarrow \tfrac{1}{2}\sqrt{2} - x = -\tfrac{1}{2}\sqrt{2} + y \Rightarrow x + y = \sqrt{2} \tag{87}$$

and

$$\mathbf{y}_{\frac{5}{4}\pi} u = (-\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}u\sqrt{2}, -\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}u\sqrt{2}) \tag{88}$$

$$\Rightarrow \begin{cases} x = -\frac{1}{2}\sqrt{2} + \frac{1}{2}u\sqrt{2} \\ y = -\frac{1}{2}\sqrt{2} - \frac{1}{2}u\sqrt{2} \end{cases} \Rightarrow \begin{cases} x\sqrt{2} = -1 + u \\ y\sqrt{2} = -1 - u \end{cases} \Rightarrow \begin{cases} u = 1 + x\sqrt{2} \\ u = -1 - y\sqrt{2} \end{cases} \tag{89}$$

$$\Rightarrow 1 + x\sqrt{2} = -1 - y\sqrt{2} \Rightarrow \tfrac{1}{2}\sqrt{2} + x = -\tfrac{1}{2}\sqrt{2} - y \Rightarrow x + y = -\sqrt{2} \tag{90}$$

so the solutions are $x + y = \pm\sqrt{2}$.

§11 Osculating Plane

☐ 11.1 If $\mathbf{x}', \mathbf{x}''$ are linearly dependent for every t then $\forall t : \mathbf{x}''t = \lambda t \cdot \mathbf{x}'t$. Write $\mathbf{w} = \mathbf{x}' \Rightarrow \mathbf{x}'' = \mathbf{w}'$, so $\mathbf{w}'t = \lambda t \cdot \mathbf{w}t$ and $w_i't = \lambda t \cdot w_it$. Solve these as first-order linear differential equations:

$$w_i'(t) = \lambda(t) \cdot w_i(t) \Leftrightarrow w_i'(t) - \lambda(t)w_i(t) = 0 \tag{91}$$

$$e^{\int -\lambda(t)\mathrm{d}t} \cdot w_i(t) = c_i \Rightarrow w_i(t) = c_i e^{\int \lambda(t)\mathrm{d}t} \tag{92}$$

so then

$$x_i'(t) = c_i e^{\int \lambda(t)\mathrm{d}t} \tag{93}$$

$$x_i(t) = \int x_i'(t)\mathrm{d}t = \int c_i e^{\int \lambda(t)\mathrm{d}t}\mathrm{d}t = c_i \int e^{\int \lambda(t)\mathrm{d}t}\mathrm{d}t + k_i = c_i t^*(t) + k_i \tag{94}$$

where $t^* = e^{\int \lambda(t)\mathrm{d}t}$, so

$$\mathbf{x}(t) = t^*(t) \cdot \mathbf{c} + \mathbf{k} \tag{95}$$

which is the parametric representation of a straight line.

§12 Principal Normal, Curvature, Osculating Circle

☐ 12.1 Starting from the definition:

$$\mathbf{m} = \mathbf{x} + \rho\mathbf{p} = \mathbf{x} + \rho^2\dot{\mathbf{t}} = \mathbf{x} + \rho^2\ddot{\mathbf{x}} \tag{96}$$

$$\ddot{\mathbf{x}} = \frac{\mathrm{d}\dot{\mathbf{x}}}{\mathrm{d}s} = \frac{\mathrm{d}\dot{\mathbf{x}}}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}s} \tag{97}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{x}' \cdot (\mathbf{x}' \cdot \mathbf{x}')^{-\frac{1}{2}}\right) \cdot \frac{1}{\sqrt{\mathbf{x}' \cdot \mathbf{x}'}} \tag{98}$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}' \cdot (\mathbf{x}' \cdot \mathbf{x}')^{-\frac{1}{2}} + \mathbf{x}' \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}' \cdot \mathbf{x}')^{-\frac{1}{2}}\right) \cdot (\mathbf{x}' \cdot \mathbf{x}')^{-\frac{1}{2}} \tag{99}$$

$$= \left(\mathbf{x}''(\mathbf{x} \cdot \mathbf{x}')^{-\frac{1}{2}} + \mathbf{x} \cdot -\tfrac{1}{2}(\mathbf{x}' \cdot \mathbf{x}')^{-\frac{3}{2}} \frac{\mathrm{d}(\mathbf{x}' \cdot \mathbf{x}')}{\mathrm{d}t}\right) \cdot (\mathbf{x}' \cdot \mathbf{x}')^{-\frac{1}{2}} \tag{100}$$

$$= \mathbf{x}''(\mathbf{x}' \cdot \mathbf{x}')^{-1} - \tfrac{1}{2}(\mathbf{x}' \cdot \mathbf{x}')^{-2} \cdot 2(\mathbf{x}' \cdot \mathbf{x}'') \tag{101}$$

$$= \frac{\mathbf{x}''(\mathbf{x}' \cdot \mathbf{x}') - \mathbf{x}(\mathbf{x}' \cdot \mathbf{x}'')}{(\mathbf{x}' \cdot \mathbf{x}')^2} \tag{102}$$

$$\rho^2 \overset{(12.7)}{=} \left(\frac{(\mathbf{x}' \cdot \mathbf{x}')^{\frac{3}{2}}}{\sqrt{(\mathbf{x}' \cdot \mathbf{x}')(\mathbf{x}'' \cdot \mathbf{x}'') - (\mathbf{x}' \cdot \mathbf{x}'')^2}}\right)^2 = \frac{(\mathbf{x}' \cdot \mathbf{x}')^3}{(\mathbf{x}' \cdot \mathbf{x}')(\mathbf{x}'' \cdot \mathbf{x}'') - (\mathbf{x}' \cdot \mathbf{x}'')^2} \tag{103}$$

$$\mathbf{m} = \mathbf{x} + \rho^2 \ddot{\mathbf{x}} \tag{104}$$

$$= \mathbf{x} + \frac{(\mathbf{x}' \cdot \mathbf{x}')^3}{(\mathbf{x}' \cdot \mathbf{x}')(\mathbf{x}'' \cdot \mathbf{x}'') - (\mathbf{x}' \cdot \mathbf{x}'')^2} \frac{\mathbf{x}''(\mathbf{x}' \cdot \mathbf{x}') - \mathbf{x}(\mathbf{x}' \cdot \mathbf{x}'')}{(\mathbf{x}' \cdot \mathbf{x}')^2} \tag{105}$$

$$= \mathbf{x} + \frac{(\mathbf{x}' \cdot \mathbf{x}')}{(\mathbf{x}' \cdot \mathbf{x}')(\mathbf{x}'' \cdot \mathbf{x}'') - (\mathbf{x}' \cdot \mathbf{x}'')^2} \left(\mathbf{x}''(\mathbf{x}' \cdot \mathbf{x}') - \mathbf{x}(\mathbf{x}' \cdot \mathbf{x}'')\right) \tag{106}$$

$$= \mathbf{x} + \frac{\mathbf{x}''(\mathbf{x}' \cdot \mathbf{x}')^2 - \mathbf{x}(\mathbf{x}' \cdot \mathbf{x}')(\mathbf{x}' \cdot \mathbf{x}'')}{(\mathbf{x}' \cdot \mathbf{x}')(\mathbf{x}'' \cdot \mathbf{x}'') - (\mathbf{x}' \cdot \mathbf{x}'')^2} \tag{107}$$

□ 12.2 From Example 9.1:

$$\mathbf{x}(s) = (r\cos\omega s, r\sin\omega s, \omega c s); \omega = \frac{1}{\sqrt{r^2 + c^2}} \tag{108}$$

$$\dot{\mathbf{x}}(s) = \frac{\mathrm{d}\mathbf{x}(s)}{\mathrm{d}s} = (-r\omega\sin\omega s, r\omega\cos\omega s, \omega c) \tag{109}$$

$$\ddot{\mathbf{x}}(s) = \frac{\mathrm{d}\dot{\mathbf{x}}(s)}{\mathrm{d}s} = (-r\omega^2\cos\omega s, -r\omega^2\sin\omega s, 0) \tag{110}$$

From §12 Example 2:

$$\kappa = r\omega^2 \Rightarrow \rho = \frac{1}{r\omega^2} \tag{111}$$

so then

$$\mathbf{m} = \mathbf{x} + \rho^2 \ddot{\mathbf{x}} \tag{112}$$

$$= \mathbf{x} + \left(\frac{1}{r\omega^2}\right)^2 (-r\omega^2\cos\omega s, -r\omega^2\sin\omega s, 0) \tag{113}$$

$$= \mathbf{x} + \frac{1}{r\omega^2}(-\cos\omega s, -\sin\omega s, 0) \tag{114}$$

$$= (r\cos\omega s, r\sin\omega s, \omega c s) + \frac{1}{r\omega^2}(-\cos\omega s, -\sin\omega s, 0) \tag{115}$$

$$= \left(-\frac{c^2}{r}\cos\omega s, -\frac{c^2}{r}\sin\omega s, \omega c s\right) \tag{116}$$

where $r - \frac{1}{r\omega^2} = r - \frac{r^2+c^2}{r} = \frac{r^2}{r} - \frac{r^2+c^2}{r} = -\frac{c^2}{r}$. For the cylinder prescribed by $\mathbf{r}$ to coincide, we must have

$$\frac{c^2}{r} = r \Rightarrow c^2 = r^2 \Rightarrow c = \pm\sqrt{r} \tag{117}$$

§13 Binormal. Moving Trihedron of a Curve.

☐ 13.1 If there is a one-to-one correspondence such that the tangents are parallel, then we can choose representations such that $\forall s : \mathbf{t}^*(s) = \mathbf{t}(s)$. Obviously then

$$\mathbf{p}^*(s) = \frac{\dot{\mathbf{t}}^*(s)}{|\dot{\mathbf{t}}^*(s)|} = \frac{\dot{\mathbf{t}}(s)}{|\dot{\mathbf{t}}(s)|} \tag{118}$$

and

$$\mathbf{b}^*(s) = \mathbf{t}^*(s) \times \mathbf{p}^*(s) = \mathbf{t}(s) \times \mathbf{p}(s) = \mathbf{b}(s) \tag{119}$$

§14 Torsion.

☐ 14.1 Begin by evaluating the 'dotted' components of (14.3):

$$\dot{\mathbf{x}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\mathrm{x}'}{s'} \tag{120}$$

$$\ddot{\mathbf{x}} = \frac{\mathrm{d}\dot{\mathbf{x}}}{\mathrm{d}s} = \frac{\mathrm{d}\dot{\mathbf{x}}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathbf{x}'}{s'}\right) \cdot \frac{1}{s'} = \left(\frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t} \cdot \frac{1}{s'} - \mathbf{x}'\frac{1}{s'^2}\frac{\mathrm{d}s'}{\mathrm{d}t}\right) \cdot \frac{1}{s'} = \frac{\mathbf{x}''}{s'^2} - \frac{\mathbf{x}'}{s'^3}s'' \tag{121}$$

$$\dddot{\mathbf{x}} = \frac{\mathrm{d}\ddot{\mathbf{x}}}{\mathrm{d}s} = \frac{\mathrm{d}\ddot{\mathbf{x}}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{x}''}{s'^2} - \frac{\mathbf{x}'}{s'^3}s''\right) \cdot \frac{1}{s'} \tag{122}$$

$$= \left(\frac{\mathrm{d}\mathbf{x}''}{\mathrm{d}t}\frac{1}{s'^2} - 2\mathbf{x}''\frac{1}{s'^3}\frac{\mathrm{d}s'}{\mathrm{d}t} - \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t}\frac{s''}{s'^3} + 3\mathbf{x}'\frac{1}{s'^4}\frac{\mathrm{d}s'}{\mathrm{d}t}s'' - \frac{\mathbf{x}'}{s'^3}\frac{\mathrm{d}s''}{\mathrm{d}t}\right)\frac{1}{s'} \tag{123}$$

$$= \mathbf{x}'''\frac{1}{s'^3} - 2\mathbf{x}''\frac{1}{s'^4}s'' - \mathbf{x}''\frac{s''}{s'^4} + 3\mathbf{x}'\frac{1}{s'^5}s''^2 - \mathbf{x}'\frac{1}{s'^4}s''' \tag{124}$$

$$= \frac{1}{s'^3}\mathbf{x}''' - 3\frac{s''}{s'^4}\mathbf{x}'' + \left(3\frac{s''^2}{s'^5} - \frac{s'''}{s'^4}\right)\mathbf{x}' \tag{125}$$

So using the rules of calculus of determinants:

$$|\dot{\mathbf{x}}\ \ddot{\mathbf{x}}\ \dddot{\mathbf{x}}| = \left|\frac{1}{s'}\mathbf{x}' \quad \frac{1}{s'^2}\mathbf{x}'' - A\mathbf{x}' \quad \frac{1}{s'^3}\mathbf{x}''' - B\mathbf{x}'' + C\mathbf{x}'\right| \tag{126}$$

$$= \left|\frac{1}{s'}\mathbf{x}' \quad \frac{1}{s'^2}\mathbf{x}'' \quad \frac{1}{s'^3}\mathbf{x}''' - B\mathbf{x}'' + C\mathbf{x}'\right| \tag{127}$$

$$= \left|\frac{1}{s'}\mathbf{x}' \quad \frac{1}{s'^2}\mathbf{x}'' \quad \frac{1}{s'^3}\mathbf{x}''' - B\mathbf{x}''\right| \tag{128}$$

$$= \left|\frac{1}{s'}\mathbf{x}' \quad \frac{1}{s'^2}\mathbf{x}'' \quad \frac{1}{s'^3}\mathbf{x}'''\right| = \frac{1}{s'^6}|\mathbf{x}'\ \ \mathbf{x}''\ \ \mathbf{x}'''| = \frac{|\mathbf{x}'\ \ \mathbf{x}''\ \ \mathbf{x}'''|}{(\mathbf{x}'\cdot\mathbf{x}')^3} \tag{129}$$

Also

$$\frac{1}{\ddot{\mathbf{x}}\cdot\ddot{\mathbf{x}}} \overset{(12.3)}{=} \frac{1}{\kappa^2} \overset{(12.7)}{=} \frac{(\mathbf{x}'\cdot\mathbf{x}')^3}{(\mathbf{x}'\cdot\mathbf{x}')(\mathbf{x}''\cdot\mathbf{x}'') - (\mathbf{x}'\cdot\mathbf{x}'')^2} \tag{130}$$

so that

$$\tau = \frac{|\dot{\mathbf{x}}\ \ddot{\mathbf{x}}\ \dddot{\mathbf{x}}|}{\ddot{\mathbf{x}} \cdot \ddot{\mathbf{x}}} = \frac{|\mathbf{x}'\ \mathbf{x}''\ \mathbf{x}'''|}{(\mathbf{x}' \cdot \mathbf{x}')(\mathbf{x}'' \cdot \mathbf{x}'') - (\mathbf{x}' \cdot \mathbf{x}'')^2} \tag{131}$$

☐ 14.2 Compute the components of (14.4):

$$\mathbf{x} = (t, \tfrac{1}{2}t^2, \tfrac{1}{6}t^3) \tag{132}$$

$$\mathbf{x}' = (1, t, \tfrac{1}{2}t^2); \quad \mathbf{x}'' = (0, 1, t); \quad \mathbf{x}''' = (0, 0, 1) \tag{133}$$

$$|\mathbf{x}'\ \mathbf{x}''\ \mathbf{x}'''| = \begin{vmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{1}{2}t^2 & t & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 \\ t & 1 \end{vmatrix} = 1 \tag{134}$$

$$\mathbf{x}' \times \mathbf{x}'' = \begin{bmatrix} 1 \\ t \\ \frac{1}{2}t^2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} = \begin{bmatrix} t^2 - \frac{1}{2}t^2 \\ -(t-0) \\ 1-0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t^2 \\ -t \\ 1 \end{bmatrix} \tag{135}$$

$$(\mathbf{x}' \times \mathbf{x}'') \cdot (\mathbf{x}' \times \mathbf{x}'') = (\tfrac{1}{2}t^2)^2 + (-t)^2 + 1 = \tfrac{1}{4}t^4 + t^2 + 1 = (1 + \tfrac{1}{2}t^2)^2 \tag{136}$$

So

$$\tau = \frac{|\mathbf{x}'\ \mathbf{x}''\ \mathbf{x}'''|}{(\mathbf{x}' \cdot \mathbf{x})(\mathbf{x}'' \cdot \mathbf{x}'') - (\mathbf{x}' \cdot \mathbf{x}'')^2} = \frac{1}{(1 + \frac{1}{2}t^2)^2} \tag{137}$$

§15 Formulae of Frenet.

☐ 15.1 Write the vectors of the trihedron as:

$$\mathbf{a}_0 = \mathbf{t}; \quad \mathbf{a}_1 = \mathbf{p}; \quad \mathbf{a}_2 = \mathbf{b} \tag{138}$$

so that the equations of Frenet can be written as

$$\dot{\mathbf{a}}_i = +_j\ c_{ij}\mathbf{a}_j \tag{139}$$

Now, since $\mathbf{t}, \mathbf{p}, \mathbf{b}$ are mutually orthogonal

$$\forall i, j : \mathbf{a}_i \cdot \mathbf{a}_j = \delta_{ij} \tag{140}$$

Differentiating:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{a}_i \cdot \mathbf{a}_j = \frac{\mathrm{d}}{\mathrm{d}s} \Rightarrow \ \dot{\mathbf{a}}_i \cdot \mathbf{a}_j + \mathbf{a}_i \cdot \dot{\mathbf{a}}_j = 0 \Rightarrow \ \dot{\mathbf{a}}_i \cdot \mathbf{a}_j = -\mathbf{a}_i \cdot \dot{\mathbf{a}}_j \tag{141}$$

Substituting the Frenet equations:

$$\dot{\mathbf{a}}_i \cdot \mathbf{a}_k = (+_j\ c_{ij}\mathbf{a}_j) \cdot \mathbf{a}_k = +_j\ c_{ij}\,\mathbf{a}_j \cdot \mathbf{a}_k = +_j\ c_{ij}\delta_{jk} = c_{ik} \tag{142}$$

so

$$\dot{\mathbf{a}}_i = \mathbf{a}_j = -\mathbf{a}_i \cdot \dot{\mathbf{a}}_j \Rightarrow \ c_{ij} = -c_{ji} \tag{143}$$

so $C = [_{ij}\ c_{ij}]$ is skew-symmetric. Therefore, from $\dot{\mathbf{t}} = \kappa\mathbf{p}, \dot{\mathbf{b}} = -\tau\mathbf{p}$:

$$C = \begin{bmatrix} 0 & \kappa & 0 \\ & & \\ 0 & -\tau & 0 \end{bmatrix} \Rightarrow \ \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \tag{144}$$

§16 Motion of the Trihedron. Vector of Darboux.

☐ 16.1 The vector of Darboux has constant direction iff it is linearly dependent on its derivative:

$$\mathbf{d} \times \dot{\mathbf{d}} = 0 \tag{145}$$

Evaluate the derivative:

$$\begin{aligned}
\dot{\mathbf{d}} &= \frac{\mathrm{d}}{\mathrm{d}s}(\tau\mathbf{t} + \kappa\mathbf{b}) = \frac{\mathrm{d}}{\mathrm{d}s}\tau\mathbf{t} + \frac{\mathrm{d}}{\mathrm{d}s}\kappa\mathbf{b} & (146)\\
&= \dot{\tau}\mathbf{t} + \tau\dot{\mathbf{t}} + \dot{\kappa}\mathbf{b} + \kappa\dot{\mathbf{b}} = \dot{\tau}\mathbf{t} + \tau\kappa\mathbf{p} + \dot{\kappa}\mathbf{b} - \kappa\tau\mathbf{p} & (147)\\
&= \dot{\tau}\mathbf{t} + \dot{\kappa}\mathbf{b} & (148)
\end{aligned}$$

So then

$$\begin{aligned}
\mathbf{d} \times \dot{\mathbf{d}} &= (\tau\mathbf{t} + \kappa\mathbf{b}) \times (\dot{\tau}\mathbf{t} + \dot{\kappa}\mathbf{b}) & (149)\\
&= \tau\mathbf{t} \times (\dot{\tau}\mathbf{t} \times \dot{\kappa}\mathbf{b}) + \kappa\mathbf{b} \times (\dot{\tau}\mathbf{t} + \dot{\kappa}\mathbf{b}) & (150)\\
&= \tau\mathbf{t} \times \dot{\kappa}\mathbf{b} + \kappa\mathbf{b} \times \dot{\tau}\mathbf{t} & (151)
\end{aligned}$$

Since $\mathbf{b} = \mathbf{t} \times \mathbf{p}$:

$$\mathbf{b} \times \mathbf{t} = (\mathbf{t} \times \mathbf{p}) \times \mathbf{t} \overset{(5.12)}{=} (\mathbf{t} \cdot \mathbf{t})\mathbf{p} - (\mathbf{p} \cdot \mathbf{t})\mathbf{t} = 1 \cdot \mathbf{p} - 0 \cdot \mathbf{t} = \mathbf{p} \tag{152}$$

so

$$\mathbf{d} \times \dot{\mathbf{d}} = (\kappa\dot{\tau} - \dot{\kappa}\tau)\mathbf{p} \tag{153}$$

so that when these are linearly dependent $\kappa\dot{\tau} - \dot{\kappa}\tau = 0$. Now since

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\tau}{\kappa} = \dot{\tau}\frac{1}{\kappa} - \tau\frac{1}{\kappa^2}\dot{\kappa} = \frac{1}{\kappa^2}(\dot{\tau}\kappa - \tau\dot{\kappa}) \tag{154}$$

this can be written as

$$\kappa^2\frac{\mathrm{d}}{\mathrm{d}s}\frac{\tau}{\kappa} = 0 \tag{155}$$

which by Theorem 15.1 are the general helices.

§17 Spherical Images of a Curve.

☐ 17.1 If $\mathbf{t}$ is constant then

$$\mathbf{x}(s) = \mathbf{x}(0) + \int_0^s \dot{\mathbf{x}}(s)\ \mathrm{d}s = \mathbf{x}(0) + \int_0^s \mathbf{t}\ \mathrm{d}s = \mathbf{x}(0) + s\mathbf{t} \tag{156}$$

which is a straight line. If $\mathbf{b}$ is constant then

$$\dot{\mathbf{b}} = 0 \overset{(Frenet)}{\Rightarrow} -\tau\mathbf{p} = 0 \Rightarrow \ \tau = 0 \vee \mathbf{p} = 0 \tag{157}$$

In the first case the curve is planar. In the second case:

$$\mathbf{p} = 0 \Rightarrow \ \dot{\mathbf{t}} = \kappa\mathbf{p} = 0 \tag{158}$$

so $\mathbf{t}$ is constant, which again implies a straight line and a planar curve. If a spherical image is a closed curve, the corresponding generating vector $\mathbf{t}, \mathbf{p}, \mathbf{b}$ is periodic.

☐ 17.2 From Exercise 12.2:

$$\mathbf{t} = \dot{\mathbf{x}} = (-r\omega \sin \omega s, \;\; r\omega \cos \omega s, \;\; \omega c) \tag{159}$$

$$\mathbf{p} = \rho \ddot{\mathbf{x}} = \frac{1}{r\omega^2}(-r\omega^2 \cos \omega s, \;\; r\omega^2 \sin \omega s, \;\; 0) = (-\cos \omega s, \;\; -\sin \omega s, \;\; 0) \tag{160}$$

$$\mathbf{b} = \mathbf{t} \times \mathbf{p} = \begin{bmatrix} -r\omega \sin \omega s \\ r\omega \cos \omega s \\ \omega c \end{bmatrix} \times \begin{bmatrix} -\cos \omega s \\ -\sin \omega s \\ 0 \end{bmatrix} = \begin{bmatrix} \omega c \sin \omega s \\ -\omega c \cos \omega s \\ r\omega \sin^2 \omega s + r\omega \cos^2 \omega s \end{bmatrix} \tag{161}$$

$$= (-\omega c \sin \omega s, \;\; -\omega c \cos \omega s, \;\; r\omega) \tag{162}$$

§18 Shape of a Curve in the Neighborhood of Any of Its Points.

☐ 18.1 Given any curve represented in a coordinate system as (18.3). Then in a sufficiently small neighborhood of $\mathbf{x}(0)$, the length h of the chord from $\mathbf{x}(0)$ to $\mathbf{x}(s)$ is:

$$h = \sqrt{x_0(s)^2 + x_1(s)^2 + x_2(s)^2} \tag{163}$$

$$= \sqrt{\left(s - \frac{\kappa_0^2}{3!}s^3 + o(s^3)\right)^2 + \left(\tfrac{1}{2}\kappa_0 s^2 + \frac{1}{3!}\dot{\kappa_0} s^3 + o(s^3)\right)^2 + \left(\frac{\kappa_0 \tau_0}{3!}s^3 + o(s^3)\right)^2} \tag{164}$$

$$= \sqrt{s^2 - \tfrac{1}{3}\kappa_0^2 s^4 + \frac{\kappa_0^4}{36}s^6 + \tfrac{1}{4}\kappa_0^2 s^4 + \tfrac{1}{6}\kappa_0 \dot{\kappa_0} s^5 + \tfrac{1}{36}\dot{\kappa_0}^2 s^6 + \frac{\kappa_0^2 \tau_0^2}{36}s^6 + o(s^6)} \tag{165}$$

$$= \sqrt{s^2 - \tfrac{1}{12}\kappa_0^2 s^4 + o(s^5)} \tag{166}$$

$$= \sqrt{s^2 - 2s\left(\tfrac{1}{24}\kappa_0^2 s^3\right) + \left(\tfrac{1}{24}\kappa_0 s^3\right)^2 + o(s^5)} \tag{167}$$

$$= \sqrt{\left(s - \tfrac{1}{24}\kappa_0^2 s^3\right)^2 + o(s^5)} = s - \tfrac{1}{24}\kappa_0^2 s^3 + o(s^?) \tag{168}$$

so the difference between the lengths of the arc and the chord is approximately

$$\left|s - \left(s - \tfrac{1}{24}\kappa_0^2 s^3\right)\right| = \tfrac{1}{24}\kappa_0^2 |s^3| \tag{169}$$

§19 Contact, Osculating Sphere.

☐ 19.1 Given the sphere $x^2 + y^2 + z^2 - r^2 = 0$ and curve $\mathbf{x}(s) = (s, r, 0)$ we have:

$$p(s) = G(\mathbf{x}(s)) = s^2 + r^2 + 0^2 - r^2 = s^2 \tag{170}$$

The point of contact follows from:

$$p(s) = 0 \Rightarrow \;\; s^2 = 0 \Rightarrow \;\; s = 0 \tag{171}$$

At this point:

$$\frac{\mathrm{d}p(s)}{\mathrm{d}s}\bigg|_{s=0} = \frac{\mathrm{d}s^2}{\mathrm{d}s}\bigg|_{s=0} = 2s|_{s=0} = 0 \tag{172}$$

$$\frac{\mathrm{d}^2 p(s)}{\mathrm{d}s^2}\Big|_{s=0} = \frac{\mathrm{d}2s}{\mathrm{d}s}\Big|_{s=0} = 2 \neq 0 \tag{173}$$

so there is a point of contact of order 1 at $(0, r, 0)$.

☐ 19.2 Given a plane defined by its normal and displacement we have

$$G(\mathbf{x}) = \mathbf{n}\cdot\mathbf{x} + d \Rightarrow \quad p(s) = G(\mathbf{x}(s)) = \mathbf{n}\cdot\mathbf{x}(s) + d \tag{174}$$

The points of contact are given by

$$order0: \quad \mathbf{n}\cdot\mathbf{x}(s_0) + d = 0 \tag{175}$$

$$order1: \quad \frac{\mathrm{d}p(s)}{\mathrm{d}s}\Big|_{s=s_0} = \mathbf{n}\cdot\dot{\mathbf{x}}(s_0) = 0 \tag{176}$$

$$order2: \quad \frac{\mathrm{d}^2 p(s)}{\mathrm{d}s^2}\Big|_{s=s_0} = \mathbf{n}\cdot\ddot{\mathbf{x}}(s_0) = 0 \tag{177}$$

Therefore $\mathbf{n} \perp \dot{\mathbf{x}}(s_0), \ddot{\mathbf{x}}(s_0)$, that is, if $\dot{\mathbf{x}}(s_0)$ and $\ddot{\mathbf{x}}(s_0)$ are linearly independent then

$$\exists\lambda \in \mathbb{R} : \mathbf{n} = \lambda\dot{\mathbf{x}}(s_0) \times \ddot{\mathbf{x}}(s_0) \tag{178}$$

So a plane with normal $\mathbf{n}$ having contact of order 2 with a curve, that plane is spawned by $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$, that is, the osculating plane at $\dot{\mathbf{x}}(s_0)$.

☐ 19.3 From (19.2), for the center $\mathbf{m}$ of the osculating sphere to be constant:

$$\dot{\mathbf{m}} = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}s}\left(\mathbf{x} + \rho\mathbf{p} + \frac{\dot{\rho}}{\tau}\mathbf{b}\right) \tag{179}$$

$$= \dot{\mathbf{x}} + \dot{\rho}\mathbf{p} + \rho\dot{\mathbf{p}} + \frac{\mathrm{d}}{\mathrm{d}s}\frac{\dot{\rho}}{\tau}\mathbf{b} - \frac{\dot{\rho}}{\tau}\dot{\mathbf{b}} \tag{180}$$

$$\overset{(Frenet)}{=} \mathbf{t} + \dot{\rho}\mathbf{p} + \rho(-\kappa\mathbf{t} + \tau\mathbf{b}) + \frac{\mathrm{d}}{\mathrm{d}s}\frac{\dot{\rho}}{\tau}\mathbf{b} - \frac{\dot{\rho}}{\tau}\tau\mathbf{p} \tag{181}$$

$$= \mathbf{t} + \dot{\rho}\mathbf{p} - \mathbf{t} + \rho\tau\mathbf{b} + \frac{\mathrm{d}}{\mathrm{d}s}\frac{\dot{\rho}}{\tau}\mathbf{b} - \dot{\rho}\mathbf{p} \tag{182}$$

$$= \left(\rho\tau + \frac{\mathrm{d}}{\mathrm{d}s}\frac{\dot{\rho}}{\tau}\right)\mathbf{b} = 0 \Rightarrow \quad \rho\tau + \frac{\mathrm{d}}{\mathrm{d}s}\frac{\dot{\rho}}{\tau} = 0 \tag{183}$$

Now it so happens that from (19.13) concerning the radius of the osculating sphere:

$$r_S = \sqrt{\rho^2 + \left(\frac{\dot{\rho}}{\tau}\right)^2} \Rightarrow \quad r_S^2 = \rho^2 + \left(\frac{\dot{\rho}}{\tau}\right)^2 \Rightarrow \tag{184}$$

$$\frac{\mathrm{d}r_S^2}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\rho^2 + \left(\frac{\dot{\rho}}{\tau}\right)^2\right) = 2\rho\dot{\rho} + 2\left(\frac{\dot{\rho}}{\tau}\right)\frac{\mathrm{d}}{\mathrm{d}s}\frac{\dot{\rho}}{\tau} = 2\frac{\dot{\rho}}{\tau}\left(\rho\tau + \frac{\mathrm{d}}{\mathrm{d}s}\frac{\dot{\rho}}{\tau}\right) = 2\frac{\dot{\rho}}{\tau}\cdot 0 = 0 \tag{185}$$

So since both the center and the radius of the osculating sphere are constant over the curve, that is, the osculating spheres coincide: the curve lies on a sphere.

§21 Examples of Curves and Their Natural Equations.

☐ 21.1 We seek an expression of curvature in terms of the 'tangent angle':

$$\alpha = \int \mathrm{d}\alpha = \int \frac{\mathrm{d}\alpha}{\mathrm{d}s}\,\mathrm{d}s \overset{(21.1)}{=} \int \kappa\,\mathrm{d}s = \int \frac{1}{\rho}\,\mathrm{d}s = \int \frac{1}{\rho}\frac{\mathrm{d}s}{\mathrm{d}\rho}\,\mathrm{d}\rho \tag{186}$$

Since $\rho = as + b \Rightarrow \frac{\mathrm{d}\rho}{\mathrm{d}s} = a$:

$$\alpha = \int_{\rho_0}^{\rho} \frac{1}{\rho}\frac{1}{a}\,\mathrm{d}\rho = \frac{1}{a}\int_{\rho_0}^{\rho}\frac{1}{\rho}\,\mathrm{d}\rho = \frac{1}{a}\left[\ln\rho\right]_{\rho_0}^{\rho} = \frac{1}{a}\ln\frac{\rho}{\rho_0} \tag{187}$$

$$\Rightarrow \quad e^{\alpha} = e^{-a}\frac{\rho}{\rho_0} \Rightarrow \quad \rho = \rho_0 e^{a\alpha} \tag{188}$$

Now

$$\frac{\mathrm{d}x}{\mathrm{d}\alpha} = \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\alpha} = \cos\alpha\cdot\rho; \qquad \frac{\mathrm{d}y}{\mathrm{d}\alpha} = \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\alpha} = \sin\alpha\cdot\rho \tag{189}$$

Treating the point as lying in the complex plane we can combine these as:

$$\frac{\mathrm{d}z}{\mathrm{d}\alpha} = \frac{\mathrm{d}x + j\mathrm{d}y}{\mathrm{d}\alpha} = \frac{\mathrm{d}x}{\mathrm{d}\alpha} + j\frac{\mathrm{d}y}{\mathrm{d}\alpha} = \rho(\cos\alpha + j\sin\alpha) = \rho e^{j\alpha} = \rho_0 e^{a\alpha}e^{j\alpha} = \rho_0 e^{(a+j)\alpha} \tag{190}$$

and integrating we therefore have

$$z = \int \mathrm{d}z = \int \frac{\mathrm{d}z}{\mathrm{d}\alpha}\,\mathrm{d}\alpha = \int \rho_0 e^{(a+j)\alpha}\,\mathrm{d}\alpha \tag{191}$$

$$= \left[\frac{\rho_0}{a+j}e^{(a+j)\alpha}\right] = \frac{\rho_0(a-j)}{(a+j)(a-j)}e^{(a+j)\alpha} = \frac{a-j}{a^2+1}\rho_0 e^{(a+j)\alpha} \tag{192}$$

Now

$$|z|^2 = zz^* = \frac{(a-j)(a+j)}{(a^2+1)^2}\rho_0^2 e^{(a+j)\alpha+(a-j)\alpha} = \frac{1}{a^2+1}\rho_0^2 e^{2a\alpha} \tag{193}$$

so the radius can be calculated as:

$$r = |z| = \frac{1}{\sqrt{a^2+1}}\rho_0 e^{a\alpha} \tag{194}$$

which is the expression of a logarithmic spiral (cf., for example, Wikipedia).

□ 21.2 Let x_1 be the parameter t on the curve, so

$$\frac{x_2}{c} - \cosh\frac{t}{c} = 0 \Rightarrow \quad x_2 = c\cosh\frac{t}{c} \tag{195}$$

Then, computing terms of the expression (12.7) of curvature:

$$\mathbf{x}' = (1,\;\; c\sinh\frac{t}{c}\cdot\frac{1}{c},\;\; 0) = (1,\;\; \sinh\frac{t}{c},\;\; 0) \tag{196}$$

$$\mathbf{x}'\cdot\mathbf{x}' = 1^2 + \sinh^2\frac{t}{c} = 1 + \left(\cosh^2\frac{t}{c} - 1\right) = \cosh^2\frac{t}{c} \tag{197}$$

$$\mathbf{x}'' = (0,\;\; \frac{1}{c}\cosh\frac{t}{c},\;\; 0) \tag{198}$$

$$\mathbf{x}''\cdot\mathbf{x}'' = \frac{1}{c^2}\cosh^2\frac{t}{c} \tag{199}$$

$$\mathbf{x}'\cdot\mathbf{x}'' = 1\cdot 0 + \frac{1}{c}\sinh\frac{t}{c}\cosh\frac{t}{c} \tag{200}$$

so that

$$\kappa = \frac{\sqrt{\cosh^2 \frac{t}{c} \cdot \frac{1}{c^2} \cosh^2 \frac{t}{c} - \left(\frac{1}{c} \sinh \frac{t}{c} \cosh \frac{t}{c}\right)^2}}{\left(\cosh^2 \frac{t}{c}\right)^{3/2}} = \frac{\sqrt{\frac{1}{c^2} \cosh^4 \frac{t}{c} - \frac{1}{c^2} \sinh^2 \frac{t}{c} \cosh^2 \frac{t}{c}}}{\cosh^3 \frac{t}{c}} \tag{201}$$

$$= \frac{\sqrt{\frac{1}{c^2} \left(\cosh^4 \frac{t}{c} - \left(\cosh^2 \frac{t}{c} - 1\right) \cosh^2 \frac{t}{c}\right)}}{\cosh^3 \frac{t}{c}} = \frac{1}{c} \frac{\sqrt{\cosh^4 \frac{t}{c} - \cosh^4 \frac{t}{c} + \cosh^2 \frac{t}{c}}}{\cosh^3 \frac{t}{c}} \tag{202}$$

$$= \frac{1}{c} \frac{\sqrt{\cosh^2 \frac{t}{c}}}{\cosh^3 \frac{t}{c}} = \frac{1}{c} \frac{\cosh \frac{t}{c}}{\cosh^3 \frac{t}{c}} = \frac{1}{c} \cosh^{-2} \frac{t}{c} \tag{203}$$

or

$$\rho = c \cosh^2 \frac{t}{c} \tag{204}$$

Now

$$s = \int \mathrm{d}s = \int \sqrt{\mathbf{x}' \cdot \mathbf{x}'}\ \mathrm{d}t = \int \sqrt{\cosh^2 \frac{t}{c}}\ \mathrm{d}t = \int \cosh \frac{t}{c}\ \mathrm{d}t = c \sinh \frac{t}{c} \tag{205}$$

so

$$s^2 = c^2 \sinh^2 \frac{t}{c} = c^2 \left(\cosh^2 \frac{t}{c} - 1\right) \Rightarrow \frac{s^2}{c^2} = \cosh^2 \frac{t}{c} - 1 \Rightarrow \cosh^2 \frac{t}{c} = \frac{s^2}{c^2} + 1 \tag{206}$$

and

$$\rho = c \left(\frac{s^2}{c^2} + 1\right) = \frac{1}{c} s^2 + c \tag{207}$$

§25 Further Remarks on the Representation of Surfaces; Examples.

□ 25.1 The representation $\mathbf{x}(u^1, u^2) = (0, u^1, u^2)$ is the yz-plane with

$$J = \begin{bmatrix} \partial x_1/\partial u^1 & \partial x_1/\partial u^2 \\ \partial x_2/\partial u^1 & \partial x_2/\partial u^2 \\ \partial x_3/\partial u^1 & \partial x_3/\partial u^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{208}$$

which has a nonvanishing determinant of order 2 and is thus itself of rank 2.

$\mathbf{x}(u^1, u^2) = (u^1 + u^2, u^1 + u^2, u^1)$ is a plane intersecting the z-axis and the line $x = y$ in the xy-plane.

$$J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{209}$$

is also of rank 2.

$\mathbf{x}(u^1, u^2) = (r \cos u^1, r \sin u^1, u^2)$ is a cylinder of radius r about the z-axis, with

$$J = \begin{bmatrix} -r \sin u^1 & 0 \\ +r \cos u^1 & 0 \\ 0 & 1 \end{bmatrix} \tag{210}$$

always has either

$$\begin{vmatrix} r\cos u^1 & 0 \\ 0 & 1 \end{vmatrix} = r\cos u^1 \neq 0 \quad \vee \quad \begin{vmatrix} -r\sin u^1 & 0 \\ 0 & 1 \end{vmatrix} = -r\sin u^1 \neq 0 \tag{211}$$

and is thus also of rank 2.

☐ 25.2 The Jacobian of the curve $\mathbf{c} = (h_1(s), h_2(s), x^3)$

$$J = \begin{bmatrix} \partial c^1/\partial s & \partial c^1/\partial x^3 \\ \partial c^2/\partial s & \partial c^2/\partial x^3 \\ \partial c^3/\partial s & \partial c^3/\partial x^3 \end{bmatrix} = \begin{bmatrix} h_1' & 0 \\ h_2' & 0 \\ 0 & 1 \end{bmatrix} \tag{212}$$

is of rank 2 as long as either $h_1' \neq 0$ or $h_2' \neq 0$, i.e., the curve is nowhere locally tangent to the z-axis.

☐ 25.3 Eliminate coordinates from the parametric equations:

$$\begin{cases} x = a\cos u^2 \cos u^1 \\ y = b\cos u^2 \sin u^1 \\ z = c\sin u^2 \end{cases} \Rightarrow \begin{cases} x^2 = a^2\cos^2 u^2 \cos^2 u^1 \\ y^2 = b^2\cos^2 u^2 \sin^2 u^1 \\ z^2 = c^2 \sin^2 u^2 \end{cases} \Rightarrow \overset{a,b,c\neq 0}{=} \begin{cases} x^2/a^2 = \cos^2 u^2 \cos^2 u^1 \\ y^2/b^2 = \cos^2 u^2 \sin^2 u^1 \\ z^2/c^2 = \sin^2 u^2 \end{cases} \tag{213}$$

so then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 u^2(\cos^2 u^1 + \sin^2 u^1) = \cos^2 u^2 \tag{214}$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \cos^2 u^2 + \sin^2 u^2 = 1 \tag{215}$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \tag{216}$$

$$\begin{cases} x = au^1 \cos u^2 \\ y = bu^1 \sin u^2 \\ z = (u^1)^2 \end{cases} \Rightarrow \begin{cases} x^2 = a^2(u^1)^2 \cos^2 u^2 \\ y^2 = b^2(u^1)^2 \sin^2 u^2 \\ z = (u^1)^2 \end{cases} \Rightarrow \begin{cases} x^2/a^2 = (u^1)^2 \cos^2 u^2 \\ y^2/b^2 = (u^1)^2 \sin^2 u^2 \\ z = (u^1)^2 \end{cases} \tag{217}$$

so

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = (u^1)^2 \left(\cos^2 u^2 + \sin^2 u^2\right) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0 \tag{218}$$

$$\begin{cases} x = au^1 \cosh u^2 \\ y = bu^1 \sinh u^2 \\ z = (u^1)^2 \end{cases} \Rightarrow \begin{cases} x^2 = a^2(u^1)^2 \cosh^2 u^2 \\ y^2 = b^2(u^1)^2 \sinh^2 u^2 \\ z = (u^1)^2 \end{cases} \Rightarrow \begin{cases} x^2/a^2 = (u^1)^2 \cosh^2 u^2 \\ y^2/b^2 = (u^1)^2 \sinh^2 u^2 \\ z = (u^1)^2 \end{cases} \tag{219}$$

so

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = (u^1)^2(\cosh^2 u^1 - \sinh^2 u^2) = (u^1)^2 \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0 \tag{220}$$

$$\begin{cases} x = a\sinh u^1 \cos u^2 \\ y = b\sinh u^1 \sin u^2 \\ z = c\cosh u^1 \end{cases} \Rightarrow \begin{cases} x^2 = a^2\sinh^2 u^1 \cos^2 u^2 \\ y^2 = b^2 \sinh^2 u^1 \sin^2 u^2 \\ z^2 = c^2\cosh^2 u^1 \end{cases} \Rightarrow \begin{cases} x^2/a^2 = \sinh^2 u^1 \cos^2 u^2 \\ y^2/b^2 = \sinh^2 u^1 \sin^2 u^2 \\ z^2/c^2 = \cosh^2 u^1 \end{cases} \tag{221}$$

so

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \sinh^2 u^1 - \cosh^2 u^1 = -1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0 \tag{222}$$

§27 First Fundamental Form. Concept of Riemannian Geometry. Summation Convention.

□ 27.1 From:

$$\mathbf{x}(u^r, u^\theta) = (r\cos\theta, r\sin\theta, 0) \tag{223}$$

$$\mathbf{x}_r = \frac{\partial \mathbf{x}}{\partial r} = (\cos\theta, \sin\theta, 0); \quad \mathbf{x}_\theta = \frac{\partial \mathbf{x}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0) \tag{224}$$

$$g_{rr} = \cos^2\theta + \sin^2\theta = 1 \tag{225}$$

$$g_{\theta\theta} = r^2\sin^2\theta + r^2\cos^2\theta = r^2 \tag{226}$$

$$g_{r\theta} = g_{\theta r} = -r\sin\theta\cos\theta + r\sin\theta\cos\theta = 0 \tag{227}$$

so

$$(\mathrm{d}s)^2 = g_{\alpha\beta}\mathrm{d}u^\alpha \mathrm{d}u^\beta = (\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2 \tag{228}$$

□ 27.2 From:

$$\mathbf{x}(u^1, u^2) = (h_1(u^1), h_2(u^1), u^2) \tag{229}$$

$$\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial u^1} = (h_1'(u^1), h_2'(u^1), 0); \quad \mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial u^2} = (0, 0, 1) \tag{230}$$

$$g_{11} = h_1'^2(u^1) + h_2'^2(u^1) \tag{231}$$

$$g_{22} = 1 \tag{232}$$

$$g_{12} = g_{21} = 0 \tag{233}$$

$$(\mathrm{d}s)^2 = g_{\alpha\beta}\mathrm{d}u^\alpha \mathrm{d}u^\beta = \left(h_1'^2(u^1) + h_2'^2(u^1)\right)(\mathrm{d}u^1)^2 + (\mathrm{d}u^2)^2 \tag{234}$$

Revolve a point about the x_3-axis by setting $h_1(u^1) = r\cos u^1$ and $h_2(u^1) = r\sin u^1$, so:

$$h_1'(u^1) = -r\sin u^1; \quad h_2'(u^1) = r\cos u^1 \tag{235}$$

$$\Rightarrow h_1'^2(u^1) + h_2'^2(u^1) = r^2\sin^2 u^1 + r^2\cos^2 u^1 = r^2 \tag{236}$$

and

$$(\mathrm{d}s)^2 = r^2(\mathrm{d}u^1)^2 + (\mathrm{d}u^2)^2 \tag{237}$$

□ 27.3 So $x_{1,2}$ are the parameters on the surface $(x^1, x^2, F(x^1, x^2))$:

$$\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial x^1} = (1, 0, \frac{\partial F}{\partial x^1}); \quad \mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial x^2} = (0, 1, \frac{\partial F}{\partial x^2}) \tag{238}$$

$$g_{11} = 1 + \left(\frac{\partial F}{\partial x^1}\right)^2 = 1 + F_1^2 \tag{239}$$

$$g_{22} = 1 + \left(\frac{\partial F}{\partial x^2}\right)^2 = 1 + F_2^2 \tag{240}$$

$$g_{12} = g_{21} = \frac{\partial F}{\partial x^1}\frac{\partial F}{\partial x^2} = F_1F_2 \tag{241}$$

so

$$(\mathrm{d}s)^2 = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta = (1+F_1^2)(\mathrm{d}x^1)^2 + 2F_1F_2\mathrm{d}x^1\mathrm{d}x^2 + (1+F_2^2)(\mathrm{d}x^2)^2 \tag{242}$$

or

$$(\mathrm{d}s)^2 = \left(1+\left(\frac{\partial x^3}{\partial x^1}\right)^2\right)(\mathrm{d}x^1)^2 + 2\frac{\partial x^3}{\partial x^1}\frac{\partial x^3}{\partial x^2}\mathrm{d}x^1\mathrm{d}x^2 + \left(1+\left(\frac{\partial x^3}{\partial x^2}\right)^2\right)(\mathrm{d}x^2)^2 \tag{243}$$

Question: Why does the book continue to use subscripts for x_α instead of superscripts like u^α?

☐ 27.4 From the expression of Cartesian coordinates:

$$\mathbf{x} = (x_1, x_2, x_3) \tag{244}$$

$$\mathbf{x}_\alpha = \frac{\partial \mathbf{x}}{\partial x_\alpha} = (_i\ \ \delta_{i\alpha}) \tag{245}$$

$$g_{\alpha\beta} = +_i\ \ \delta_{i\alpha}\delta_{i\beta} = \delta_{\alpha\beta} \tag{246}$$

$$(\mathrm{d}s)^2 = +_\alpha +_\beta\, g_{\alpha\beta}\ \ \mathrm{d}x_\alpha\mathrm{d}x_\beta = +_\alpha +_\beta\, \delta_{\alpha\beta}\ \ \mathrm{d}x_\alpha\mathrm{d}x_\beta = +_\alpha(\mathrm{d}x_\alpha)^2 = (\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 + (\mathrm{d}x_3)^2 \tag{247}$$

For spherical coordinates:

$$\mathbf{x} = (r\cos\theta\cos\phi, r\cos\theta\sin\phi, r\sin\theta) \tag{248}$$

$$\mathbf{x}_\theta = \frac{\partial \mathbf{x}}{\partial \theta} = (-r\sin\theta\cos\phi, -r\sin\theta\sin\phi, r\cos\theta) \tag{249}$$

$$\mathbf{x}_\phi = \frac{\partial \mathbf{x}}{\partial \phi} = (-r\cos\theta\sin\phi, r\cos\theta\cos\phi, 0) \tag{250}$$

$$\mathbf{x}_r = \frac{\partial \mathbf{x}}{\partial r} = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta) \tag{251}$$

$$g_{\theta\theta} = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta = r^2\sin^2\theta + r^2\cos^2\theta = r^2 \tag{252}$$

$$g_{\phi\phi} = r^2\cos^2\theta\sin^2\phi + r^2\cos^2\theta\cos^2\phi = r^2\cos^2\theta \tag{253}$$

$$g_{rr} = \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta = \cos^2\theta + \sin^2\theta = 1 \tag{254}$$

$$g_{\theta\phi} = g_{\phi\theta} = r^2\sin\theta\cos\theta\sin\phi\cos\phi - r^2\sin\theta\cos\theta\sin\phi\cos\phi = 0 \tag{255}$$

$$g_{\theta r} = g_{r\theta} = -r\sin\theta\cos\theta\cos^2\phi - r\sin\theta\cos\theta\sin^2\phi + r\sin\theta\cos\theta \tag{256}$$

$$= -r\sin\theta\cos\theta + r\sin\theta\cos\theta = 0 \tag{257}$$

$$g_{\phi r} = g_{r\phi} = -r\cos^2\theta\sin\phi\cos\phi + r\cos^2\theta\sin\phi\cos\phi = 0 \tag{258}$$

$$(\mathrm{d}s)^2 = +_{\alpha,\beta}\ \ g_{\alpha\beta}\ \ \mathrm{d}x_\alpha\mathrm{d}x_\beta = r^2(\mathrm{d}\theta)^2 + r^2\cos^2\theta(\mathrm{d}\phi)^2 + (\mathrm{d}r)^2 \tag{259}$$

§28 Properties of the First Fundamental Form.

△ 28.1 Prove that the first fundamental form is positive definite by developing:

$$g_{11} = \frac{\partial \mathbf{x}}{\partial u^1} \cdot \frac{\partial \mathbf{x}}{\partial u^1} = \begin{bmatrix} \partial x_1/\partial u^1 \\ \partial x_2/\partial u^1 \\ \partial x_3/\partial u^1 \end{bmatrix} \cdot \begin{bmatrix} \partial x_1/\partial u^1 \\ \partial x_2/\partial u^1 \\ \partial x_3/\partial u^1 \end{bmatrix} = \left(\frac{\partial x_1}{\partial u^1}\right)^2 + \left(\frac{\partial x_2}{\partial u^1}\right)^2 + \left(\frac{\partial x_3}{\partial u^1}\right)^2 \tag{260}$$

$$g_{22} = \frac{\partial \mathbf{x}}{\partial u^2} \cdot \frac{\partial \mathbf{x}}{\partial u^2} = \begin{bmatrix} \partial x_1/\partial u^2 \\ \partial x_2/\partial u^2 \\ \partial x_3/\partial u^2 \end{bmatrix} \cdot \begin{bmatrix} \partial x_1/\partial u^2 \\ \partial x_2/\partial u^2 \\ \partial x_3/\partial u^2 \end{bmatrix} = \left(\frac{\partial x_1}{\partial u^2}\right)^2 + \left(\frac{\partial x_2}{\partial u^2}\right)^2 + \left(\frac{\partial x_3}{\partial u^2}\right)^2 \tag{261}$$

$$g_{12} = g_{21} = \frac{\partial \mathbf{x}}{\partial u^1} \cdot \frac{\partial \mathbf{x}}{\partial u^2} = \begin{bmatrix} \partial x_1/\partial u^1 \\ \partial x_2/\partial u^1 \\ \partial x_3/\partial u^1 \end{bmatrix} \cdot \begin{bmatrix} \partial x_1/\partial u^2 \\ \partial x_2/\partial u^2 \\ \partial x_3/\partial u^2 \end{bmatrix} = \frac{\partial x_1}{\partial u^1}\frac{\partial x_1}{\partial u^2} + \frac{\partial x_2}{\partial u^1}\frac{\partial x_2}{\partial u^2} + \frac{\partial x_3}{\partial u^1}\frac{\partial x_3}{\partial u^2} \tag{262}$$

Then

$$g_{11}g_{22} - g_{12}g_{21} = \left(\left(\frac{\partial x_1}{\partial u^1}\right)^2 + \left(\frac{\partial x_2}{\partial u^1}\right)^2 + \left(\frac{\partial x_3}{\partial u^1}\right)^2\right) \cdot \left(\left(\frac{\partial x_1}{\partial u^2}\right)^2 + \left(\frac{\partial x_2}{\partial u^2}\right)^2 + \left(\frac{\partial x_3}{\partial u^2}\right)^2\right) \tag{263}$$

$$- \left(\frac{\partial x_1}{\partial u^1}\frac{\partial x_1}{\partial u^2} + \frac{\partial x_2}{\partial u^1}\frac{\partial x_2}{\partial u^2} + \frac{\partial x_3}{\partial u^1}\frac{\partial x_3}{\partial u^2}\right) \cdot \left(\frac{\partial x_1}{\partial u^2}\frac{\partial x_1}{\partial u^1} + \frac{\partial x_2}{\partial u^2}\frac{\partial x_2}{\partial u^1} + \frac{\partial x_3}{\partial u^2}\frac{\partial x_3}{\partial u^1}\right) \tag{264}$$

$$= \left(\frac{\partial x_1}{\partial u_1}\right)^2 \left(\frac{\partial x_1}{\partial u_2}\right)^2 + \left(\frac{\partial x_1}{\partial u_1}\right)^2 \left(\frac{\partial x_2}{\partial u_2}\right)^2 + \left(\frac{\partial x_1}{\partial u_1}\right)^2 \left(\frac{\partial x_3}{\partial u_2}\right)^2 \tag{265}$$

$$+ \left(\frac{\partial x_2}{\partial u_1}\right)^2 \left(\frac{\partial x_1}{\partial u_2}\right)^2 + \left(\frac{\partial x_2}{\partial u_1}\right)^2 \left(\frac{\partial x_2}{\partial u_2}\right)^2 + \left(\frac{\partial x_2}{\partial u_1}\right)^2 \left(\frac{\partial x_3}{\partial u_2}\right)^2 \tag{266}$$

$$+ \left(\frac{\partial x_3}{\partial u_1}\right)^2 \left(\frac{\partial x_1}{\partial u_2}\right)^2 + \left(\frac{\partial x_3}{\partial u_1}\right)^2 \left(\frac{\partial x_2}{\partial u_2}\right)^2 + \left(\frac{\partial x_3}{\partial u_1}\right)^2 \left(\frac{\partial x_3}{\partial u_2}\right)^2 \tag{267}$$

$$- \frac{\partial x_1}{\partial u^1}\frac{\partial x_1}{\partial u^2} \cdot \frac{\partial x_1}{\partial u^1}\frac{\partial x_1}{\partial u^2} - \frac{\partial x_1}{\partial u^1}\frac{\partial x_1}{\partial u^2} \cdot \frac{\partial x_2}{\partial u^1}\frac{\partial x_2}{\partial u^2} - \frac{\partial x_1}{\partial u^1}\frac{\partial x_1}{\partial u^2} \cdot \frac{\partial x_3}{\partial u^1}\frac{\partial x_3}{\partial u^2} \tag{268}$$

$$- \frac{\partial x_2}{\partial u^1}\frac{\partial x_2}{\partial u^2} \cdot \frac{\partial x_1}{\partial u^1}\frac{\partial x_1}{\partial u^2} - \frac{\partial x_2}{\partial u^1}\frac{\partial x_2}{\partial u^2} \cdot \frac{\partial x_2}{\partial u^1}\frac{\partial x_2}{\partial u^2} - \frac{\partial x_2}{\partial u^1}\frac{\partial x_2}{\partial u^2} \cdot \frac{\partial x_3}{\partial u^1}\frac{\partial x_3}{\partial u^2} \tag{269}$$

$$- \frac{\partial x_3}{\partial u^1}\frac{\partial x_3}{\partial u^2} \cdot \frac{\partial x_1}{\partial u^1}\frac{\partial x_1}{\partial u^2} - \frac{\partial x_3}{\partial u^1}\frac{\partial x_3}{\partial u^2} \cdot \frac{\partial x_2}{\partial u^1}\frac{\partial x_2}{\partial u^2} - \frac{\partial x_3}{\partial u^1}\frac{\partial x_3}{\partial u^2} \cdot \frac{\partial x_3}{\partial u^1}\frac{\partial x_3}{\partial u^2} \tag{270}$$

$$= \left(\frac{\partial x_1}{\partial u^1}\frac{\partial x_2}{\partial u^2} - \frac{\partial x_2}{\partial u^1}\frac{\partial x_1}{\partial u^2}\right)^2 + \left(\frac{\partial x_1}{\partial u^1}\frac{\partial x_3}{\partial u^2} - \frac{\partial x_3}{\partial u^1}\frac{\partial x_1}{\partial u^2}\right)^2 + \left(\frac{\partial x_2}{\partial u^1}\frac{\partial x_3}{\partial u^2} - \frac{\partial x_3}{\partial u^1}\frac{\partial x_2}{\partial u^2}\right)^2 \tag{271}$$

which is the sum of the squares of the three subdeterminants of the Jacobian. Since J is of rank 2, at least one of these must be nonzero, so this sum is positive and $g_{11}g_{22} - g_{12}g_{21} > 0$.

△ 28.2 Develop the expression of the discriminant:

$$\bar{g} = \bar{g}_{11}\bar{g}_{22} - \bar{g}_{12}\bar{g}_{21} \tag{272}$$

$$= \frac{\partial u^\alpha}{\partial \bar{u}^1}\frac{\partial u^\beta}{\partial \bar{u}^1} g_{\alpha\beta} \frac{\partial u^\mu}{\partial \bar{u}^2}\frac{\partial u^\nu}{\partial \bar{u}^2} g_{\mu\nu} - \frac{\partial u^\alpha}{\partial \bar{u}^1}\frac{\partial u^\beta}{\partial \bar{u}^2} g_{\alpha\beta} \frac{\partial u^\mu}{\partial \bar{u}^2}\frac{\partial u^\nu}{\partial \bar{u}^1} g_{\mu\nu} \tag{273}$$

$$= \frac{\partial u^\alpha}{\partial \bar{u}^1}\frac{\partial u^\beta}{\partial \bar{u}^1}\frac{\partial u^\mu}{\partial \bar{u}^2}\frac{\partial u^\nu}{\partial \bar{u}^2} g_{\alpha\beta}g_{\mu\nu} - \frac{\partial u^\alpha}{\partial \bar{u}^1}\frac{\partial u^\beta}{\partial \bar{u}^2}\frac{\partial u^\mu}{\partial \bar{u}^2}\frac{\partial u^\nu}{\partial \bar{u}^1} g_{\alpha\beta}g_{\mu\nu} \tag{274}$$

$$= \left(\frac{\partial u^\alpha}{\partial \bar{u}^1}\frac{\partial u^\beta}{\partial \bar{u}^1}\frac{\partial u^\mu}{\partial \bar{u}^2}\frac{\partial u^\nu}{\partial \bar{u}^2} - \frac{\partial u^\alpha}{\partial \bar{u}^1}\frac{\partial u^\beta}{\partial \bar{u}^2}\frac{\partial u^\mu}{\partial \bar{u}^2}\frac{\partial u^\nu}{\partial \bar{u}^1}\right) g_{\alpha\beta}g_{\mu\nu} \tag{275}$$

$$= \frac{\partial u^\alpha}{\partial \bar{u}^1}\frac{\partial u^\mu}{\partial \bar{u}^2}\left(\frac{\partial u^\beta}{\partial \bar{u}^1}\frac{\partial u^\nu}{\partial \bar{u}^2} - \frac{\partial u^\beta}{\partial \bar{u}^2}\frac{\partial u^\nu}{\partial \bar{u}^1}\right) g_{\alpha\beta}g_{\mu\nu} \tag{276}$$

The parenthesized expression vanishes if $\beta = \nu$, so

$$= \frac{\partial u^\alpha}{\partial \bar{u}^1}\frac{\partial u^\mu}{\partial \bar{u}^2}\left[\left(\frac{\partial u^1}{\partial \bar{u}^1}\frac{\partial u^2}{\partial \bar{u}^2} - \frac{\partial u^1}{\partial \bar{u}^2}\frac{\partial u^2}{\partial \bar{u}^1}\right) g_{\alpha 1} g_{\mu 2} + \left(\frac{\partial u^2}{\partial \bar{u}^1}\frac{\partial u^1}{\partial \bar{u}^2} - \frac{\partial u^2}{\partial \bar{u}^2}\frac{\partial u^1}{\partial \bar{u}^1}\right) g_{\alpha 2} g_{\mu 1}\right] \tag{277}$$

$$= \frac{\partial u^\alpha}{\partial \bar{u}^1}\frac{\partial u^\mu}{\partial \bar{u}^2}\left[\left(\frac{\partial u^1}{\partial \bar{u}^1}\frac{\partial u^2}{\partial \bar{u}^2} - \frac{\partial u^1}{\partial \bar{u}^2}\frac{\partial u^2}{\partial \bar{u}^1}\right) g_{\alpha 1} g_{\mu 2} - \left(\frac{\partial u^1}{\partial \bar{u}^1}\frac{\partial u^2}{\partial \bar{u}^2} - \frac{\partial u^1}{\partial \bar{u}^2}\frac{\partial u^2}{\partial \bar{u}^1}\right) g_{\alpha 2} g_{\mu 1}\right] \tag{278}$$

$$= \frac{\partial u^\alpha}{\partial \bar{u}^1}\frac{\partial u^\mu}{\partial \bar{u}^2}\left(\frac{\partial u^1}{\partial \bar{u}^1}\frac{\partial u^2}{\partial \bar{u}^2} - \frac{\partial u^1}{\partial \bar{u}^2}\frac{\partial u^2}{\partial \bar{u}^1}\right)(g_{\alpha 1} g_{\mu 2} - g_{\alpha 2} g_{\mu 1}) \tag{279}$$

The rightmost parenthesized expression vanishes if $\alpha = \mu$, so

$$= \left(\frac{\partial u^1}{\partial \bar{u}^1}\frac{\partial u^2}{\partial \bar{u}^2} - \frac{\partial u^1}{\partial \bar{u}^2}\frac{\partial u^2}{\partial \bar{u}^1}\right)\left[\frac{\partial u^1}{\partial \bar{u}^1}\frac{\partial u^2}{\partial \bar{u}^2}(g_{11} g_{22} - g_{12} g_{21}) + \frac{\partial u^2}{\partial \bar{u}^1}\frac{\partial u^1}{\partial \bar{u}^2}(g_{21} g_{12} - g_{22} g_{11})\right] \tag{280}$$

$$= \left(\frac{\partial u^1}{\partial \bar{u}^1}\frac{\partial u^2}{\partial \bar{u}^2} - \frac{\partial u^1}{\partial \bar{u}^2}\frac{\partial u^2}{\partial \bar{u}^1}\right)\left(\frac{\partial u^1}{\partial \bar{u}^1}\frac{\partial u^2}{\partial \bar{u}^2} - \frac{\partial u^2}{\partial \bar{u}^1}\frac{\partial u^1}{\partial \bar{u}^2}\right)(g_{11} g_{22} - g_{12} g_{21}) \tag{281}$$

$$= \left(\frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)}\right)^2 (g_{11} g_{22} - g_{12} g_{21}). \tag{282}$$

□ 28.1 A different local parametric representation than the one given in the Answers:

$$\mathbf{x} = (r(z)\cos\phi, r(z)\sin\phi, z) \tag{283}$$

is allowable everywhere except where $\mathrm{d}r/\mathrm{d}z = \infty$.

$$\mathbf{x}_\phi = (-r(z)\sin\phi,\ \ r(z)\cos\phi,\ \ 0) \tag{284}$$

$$\mathbf{x}_z = (r'(z)\cos\phi,\ \ r'(z)\sin\phi,\ \ 1) \tag{285}$$

So the components of the metric tensor:

$$g_{\phi\phi} = \mathbf{x}_\phi \cdot \mathbf{x}_\phi = r^2(z)\sin^2\phi + r^2(z)\cos^2\phi = r^2(z) \tag{286}$$

$$g_{zz} = \mathbf{x}_z \cdot \mathbf{x}_z = r'^2(z)\cos^2\phi + r'^2(z)\sin^2\phi + 1 = r'^2(z) + 1 \tag{287}$$

$$g_{z\phi} = g_{\phi z} = \mathbf{x}_\phi \cdot \mathbf{x}_z = -r^2(z)\sin\phi\cos\phi + r^2(z)\sin\phi\cos\phi = 0 \tag{288}$$

and

$$(\mathrm{d}s)^2 = r^2(z)(\mathrm{d}\phi)^2 + \left(r'^2(z) + 1\right)(\mathrm{d}z)^2 \tag{289}$$

□ 28.2 Again, a different local parametric representation:

$$\mathbf{x} = (r\cos\phi(z), r\sin\phi(z), z) \tag{290}$$

is allowable everywhere except where $\mathrm{d}\phi/\mathrm{d}z = \infty$.

$$\mathbf{x}_r = (\cos\phi(z),\ \ \sin\phi(z),\ \ 0) \tag{291}$$

$$\mathbf{x}_z = (-r\sin\phi(z)\cdot\phi'(z),\ \ r\cos\phi(z)\cdot\phi'(z),\ \ 1) \tag{292}$$

So the components of the metric tensor:

$$g_r = \cos^2\phi(z) + \sin^2\phi(z) \tag{293}$$

$$g_{zz} = r^2\sin^2\phi(z)\cdot\phi'^2(z) + r^2\cos^2\phi(z)\cdot\phi'^2(z) + 1 = r^2\phi'^2(z) + 1 \tag{294}$$

$$g_{rz} = g_{zr} = -r\sin\phi(z)\cos\phi(z)\cdot\phi'(z) + r\sin\phi(z)\cos\phi(z)\cdot\phi'(z) = 0 \tag{295}$$

and

$$(\mathrm{d}s)^2 = (\mathrm{d}r)^2 + \left(r^2\phi'^2(z) + 1\right)(\mathrm{d}z)^2 \tag{296}$$

☐ 28.3 By straightforward expansion of

$$\frac{\bar{g}}{g} = \left(\frac{\partial(u^1,u^2)}{\partial(\bar{u}^1,\bar{u}^2)}\right)^2 = \begin{vmatrix} \partial u^1/\partial\bar{u}^1 & \partial u^1/\partial\bar{u}^2 \\ \partial u^2/\partial\bar{u}^1 & \partial u^2/\partial\bar{u}^2 \end{vmatrix}^2 \tag{297}$$

we get

$$\frac{\bar{\bar{g}}}{g} = \frac{\bar{g}}{g}\cdot\frac{\bar{\bar{g}}}{\bar{g}} \tag{298}$$

$$= \begin{vmatrix} \partial u^1/\partial\bar{u}^1 & \partial u^1/\partial\bar{u}^2 \\ \partial u^2/\partial\bar{u}^1 & \partial u^2/\partial\bar{u}^2 \end{vmatrix}^2 \cdot \begin{vmatrix} \partial\bar{u}^1/\partial\bar{\bar{u}}^1 & \partial\bar{u}^1/\partial\bar{\bar{u}}^2 \\ \partial\bar{u}^2/\partial\bar{\bar{u}}^1 & \partial\bar{u}^2/\partial\bar{\bar{u}}^2 \end{vmatrix}^2 \tag{299}$$

$$= \left(\begin{vmatrix} \partial u^1/\partial\bar{u}^1 & \partial u^1/\partial\bar{u}^2 \\ \partial u^2/\partial\bar{u}^1 & \partial u^2/\partial\bar{u}^2 \end{vmatrix} \cdot \begin{vmatrix} \partial\bar{u}^1/\partial\bar{\bar{u}}^1 & \partial\bar{u}^1/\partial\bar{\bar{u}}^2 \\ \partial\bar{u}^2/\partial\bar{\bar{u}}^1 & \partial\bar{u}^2/\partial\bar{\bar{u}} \end{vmatrix}\right)^2 \tag{300}$$

$$= \left(\begin{vmatrix} \partial u^1/\partial\bar{u}^\alpha\cdot\partial\bar{u}^\alpha/\partial\bar{\bar{u}}^1 & \partial u^1/\partial\bar{u}^\alpha\cdot\partial\bar{u}^\alpha/\partial\bar{\bar{u}}^2 \\ \partial u^2/\partial\bar{u}^\alpha\cdot\partial\bar{u}^\alpha/\partial\bar{\bar{u}}^1 & \partial u^2/\partial\bar{u}^\alpha\cdot\partial\bar{u}^\alpha/\partial\bar{\bar{u}}^2 \end{vmatrix}\right)^2 \tag{301}$$

$$= \begin{vmatrix} \partial u^1/\partial\bar{\bar{u}}^1 & \partial u^1/\partial\bar{\bar{u}}^2 \\ \partial u^2/\partial\bar{\bar{u}}^1 & \partial u^2/\partial\bar{\bar{u}}^2 \end{vmatrix}^2 = \left(\frac{\partial(u^1,u^2)}{\partial(\bar{\bar{u}}^1,\bar{\bar{u}}^2)}\right)^2. \tag{302}$$

☐ 28.4 This follows from the expansion of:

$$D\bar{D} = \frac{\partial(u^1,u^2)}{\partial(\bar{u}^1,\bar{u}^2)}\cdot\frac{\partial(\bar{u}^1,\bar{u}^2)}{\partial(u^1,u^2)} \tag{303}$$

$$= \begin{vmatrix} \partial u^1/\partial\bar{u}^1 & \partial u^1/\partial\bar{u}^2 \\ \partial u^2/\partial\bar{u}^1 & \partial u^2/\partial\bar{u}^2 \end{vmatrix} \cdot \begin{vmatrix} \partial\bar{u}^1/\partial u^1 & \partial\bar{u}^1/\partial u^2 \\ \partial\bar{u}^2/\partial u^1 & \partial\bar{u}^2/\partial u^2 \end{vmatrix} \tag{304}$$

$$= \begin{vmatrix} \partial u^1/\partial\bar{u}^\alpha\cdot\partial\bar{u}^\alpha/\partial u^1 & \partial u^1/\partial\bar{u}^\alpha\cdot\partial u^\alpha/\partial\bar{u}^2 \\ \partial u^2/\partial\bar{u}^\alpha\cdot\partial\bar{u}^\alpha/\partial u^1 & \partial u^2/\partial\bar{u}^\alpha\cdot\partial\bar{u}^\alpha/\partial u^2 \end{vmatrix} \tag{305}$$

$$= \begin{vmatrix} \partial u^1/\partial u^1 & \partial u^1/\partial u^2 \\ \partial u^2/\partial u^1 & \partial u^2/\partial u^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \tag{306}$$

☐ 28.5 Unsolved.

§31 Basic rules of tensor calculus.

☐ 31.1 Pair terms between upper and lower diagonals:

$$a^{\alpha\beta}b_{\alpha\beta} = +_\alpha +_\beta\, a^{\alpha\beta}b_{\alpha\beta} \tag{307}$$

$$= \left(+_{\alpha,\beta:\alpha=\beta}\; a^{\alpha\beta}b_{\alpha\beta}\right) + \left(+_{\alpha,\beta:\alpha<\beta}\; a^{\alpha\beta}b_{\alpha\beta} + a^{\beta\alpha}b_{\beta\alpha}\right) \tag{308}$$

$$= \left(+_{\alpha,\beta:\alpha=\beta}\; a^{\alpha\beta}\cdot 0\right) + \left(+_{\alpha,\beta:\alpha<\beta}\; a^{\alpha\beta}b_{\alpha\beta} - a^{\alpha\beta}b_{\alpha\beta}\right) \tag{309}$$

$$= 0 + 0 = 0. \tag{310}$$

☐ 31.2 From:

$$\bar{\bar{T}}^{\lambda_1\ldots\lambda_s}_{\kappa_1\ldots\kappa_r} = \frac{\partial\bar{u}^{\gamma_1}}{\partial\bar{\bar{u}}^{\kappa_1}}\cdots\frac{\partial\bar{u}^{\gamma_r}}{\partial\bar{\bar{u}}^{\kappa_r}}\cdot\frac{\partial\bar{\bar{u}}^{\lambda_1}}{\partial\bar{u}^{\eta_1}}\cdots\frac{\partial\bar{\bar{u}}^{\lambda_s}}{\partial\bar{u}^{\eta_s}}\,\bar{T}^{\eta_1\ldots\eta_s}_{\gamma_1\ldots\gamma_r} \tag{311}$$

$$\bar{T}^{\eta_1\ldots\eta_s}_{\gamma_1\ldots\gamma_r} = \frac{\partial u^{\alpha_1}}{\partial \bar{u}^{\gamma_1}}\cdots\frac{\partial u^{\alpha_r}}{\partial \bar{u}^{\gamma_r}}\cdot\frac{\partial \bar{u}^{\eta_1}}{\partial u^{\beta_1}}\cdots\frac{\partial \bar{u}^{\eta_s}}{\partial u^{\beta_s}}\bar{T}^{\beta_1\ldots\beta_s}_{\alpha_1\ldots\alpha_r} \tag{312}$$

we obtain:

$$\bar{\bar{T}}^{\lambda_1\ldots\lambda_s}_{\kappa_1\ldots\kappa_r} = \frac{\partial \bar{u}^{\gamma_1}}{\partial \bar{\bar{u}}^{\kappa_1}}\cdots\frac{\partial \bar{u}^{\gamma_r}}{\partial \bar{\bar{u}}^{\kappa_r}}\cdot\frac{\partial u^{\alpha_1}}{\partial \bar{u}^{\gamma_1}}\cdots\frac{\partial u^{\alpha_r}}{\partial \bar{u}^{\gamma_r}}\cdot\frac{\partial \bar{\bar{u}}^{\lambda_1}}{\partial \bar{u}^{\eta_1}}\cdots\frac{\partial \bar{\bar{u}}^{\lambda_s}}{\partial \bar{u}^{\eta_s}}\cdot\frac{\partial \bar{u}^{\eta_1}}{\partial u^{\beta_1}}\cdots\frac{\partial \bar{u}^{\eta_s}}{\partial u^{\beta_s}}T^{\beta_1\ldots\beta_s}_{\alpha_1\ldots\alpha_r} \tag{313}$$

$$= \frac{\partial u^{\alpha_1}}{\partial \bar{\bar{u}}^{\kappa_1}}\cdots\frac{\partial u^{\alpha_r}}{\partial \bar{\bar{u}}^{\kappa_r}}\cdot\frac{\partial \bar{\bar{u}}^{\lambda_1}}{\partial u^{\beta_1}}\cdots\frac{\partial \bar{\bar{u}}^{\lambda_s}}{\partial u^{\beta_s}}T^{\beta_1\ldots\beta_s}_{\alpha_1\ldots\alpha_r}. \tag{314}$$

§33 Special tensors.

☐ 33.1 By (33.2), $\epsilon^{\mu\nu} = \epsilon_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}$ so:

$$g_{\mu\sigma}\epsilon^{\mu\nu} = \epsilon_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}g_{\mu\sigma} = \epsilon_{\alpha\beta}g^{\nu\beta}\delta^{\alpha}_{\sigma} = \epsilon_{\sigma\beta}g^{\nu\beta} \tag{315}$$

$$g_{\mu\sigma}g_{\nu\tau}\epsilon^{\mu\nu} = \epsilon_{\sigma\beta}g^{\nu\beta}g_{\nu\tau} = \epsilon_{\sigma\beta}\delta^{\beta}_{\tau} = \epsilon_{\sigma\tau} \tag{316}$$

so then

$$\epsilon_{00} = \epsilon^{\alpha\beta}g_{0\alpha}g_{0\beta} = \epsilon^{01}g_{00}g_{01} + \epsilon^{10}g_{01}g_{00} = \epsilon^{01}g_{00}g_{01} - \epsilon^{01}g_{01}g_{00} = 0 \tag{317}$$

$$\epsilon_{11} = \epsilon^{\alpha\beta}g_{1\alpha}g_{1\beta} = \epsilon^{01}g_{10}g_{11} + \epsilon^{10}g_{11}g_{10} = \epsilon^{01}g_{10}g_{11} - \epsilon^{01}g_{11}g_{10} = 0 \tag{318}$$

$$\epsilon_{01} = \epsilon^{\alpha\beta}g_{0\alpha}g_{1\beta} = \epsilon^{01}g_{00}g_{11} + \epsilon^{10}g_{01}g_{10} = \frac{1}{\sqrt{g}}g_{00}g_{11} - \frac{1}{\sqrt{g}}g_{01}g_{10} = \frac{1}{\sqrt{g}}\begin{vmatrix} g_{00} & g_{10} \\ g_{01} & g_{11}\end{vmatrix} = \frac{1}{\sqrt{g}}g = \sqrt{g} \tag{319}$$

$$\epsilon_{10} = \epsilon^{\alpha\beta}g_{1\alpha}g_{0\beta} = \ldots = -\sqrt{g} \tag{320}$$

$$\tag{321}$$

☐ 33.2 From (33.4), $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\beta}$. With the result of Exercise 1 $\epsilon_{\beta\gamma} = \epsilon^{\mu\nu}g_{\mu\beta}g_{\nu\gamma}$ we have:

$$\epsilon^{\alpha\gamma}\epsilon^{\mu\nu}g_{\mu\beta}g_{\nu\gamma} = \delta^{\alpha}_{\beta} \tag{322}$$

and

$$\epsilon^{\alpha\gamma}\epsilon^{\mu\nu}g_{\mu\beta}g_{\nu\gamma}g^{\beta\kappa} = \delta^{\alpha}_{\beta}g^{\beta\kappa} \tag{323}$$

$$\epsilon^{\alpha\gamma}\epsilon^{\mu\nu}\delta^{\kappa}_{\mu}g_{\nu\gamma} = \delta^{\alpha}_{\beta}g^{\beta\kappa} \tag{324}$$

$$\epsilon^{\alpha\gamma}\epsilon^{\kappa\nu}g_{\nu\gamma} = g^{\alpha\kappa}. \tag{325}$$

☐ 33.3 Simply: $g^{\beta}_{\alpha} = g_{\alpha\gamma}\cdot g^{\gamma\beta} = \delta^{\beta}_{\alpha}$.

☐ 33.4 Expanding by terms:

$$\delta^{\beta}_{\alpha} = \delta^{0}_{0} + \delta^{1}_{1} = 1 + 1 = 2 \tag{326}$$

$$\delta^{\beta}_{\alpha}\delta^{\alpha}_{\beta} = \delta^{0}_{0}\delta^{0}_{0} + \delta^{1}_{0}\delta^{0}_{1} + \delta^{0}_{1}\delta^{1}_{0} + \delta^{1}_{1}\delta^{1}_{1} = 1 + 0 + 0 + 1 = 2 \tag{327}$$

$$\delta^{\beta}_{\alpha}\delta^{\gamma}_{\beta}\delta^{\alpha}_{\gamma} = \delta^{0}_{0}\delta^{0}_{0}\delta^{0}_{0} + \ldots + \delta^{1}_{1}\delta^{1}_{1}\delta^{1}_{1} = 1 + 0 + \ldots + 0 + 1 = 2 \tag{328}$$

§37 Remarks on the definition of area.

□ 37.1 Renaming coordinates in (25.4) we have:

$$\mathbf{x}(r,\phi) = (r\cos\phi, r\sin\phi, ar) \tag{329}$$

$$\mathbf{x}_r = \frac{\partial \mathbf{x}}{\partial r} = (\cos\phi, \sin\phi, a) \tag{330}$$

$$\mathbf{x}_\phi = \frac{\partial \mathbf{x}}{\partial \phi} = (-r\sin\phi, r\cos\phi, 0) \tag{331}$$

$$g_{rr} = \mathbf{x}_r \cdot \mathbf{x}_r = \cos^2\phi + \sin^2\phi + a^2 = a^2 + 1 \tag{332}$$

$$g_{\phi\phi} = \mathbf{x}_\phi \cdot \mathbf{x}_\phi = r^2\sin^2\phi + r^2\cos^2\phi + 0 = r^2 \tag{333}$$

$$g_{\phi r} = g_{r\phi} = \mathbf{x}_r \cdot \mathbf{x}_\phi = -r\sin\phi\cos\phi + r\sin\phi\cos\phi + 0 = 0 \tag{334}$$

so

$$g = \begin{vmatrix} a^2+1 & 0 \\ 0 & r^2 \end{vmatrix} = (a^2+1)r^2 \tag{335}$$

and then

$$A = \int\limits_{\phi=0}^{2\pi}\int\limits_{r=0}^{B/a} \sqrt{g}\ \mathrm{d}r\,\mathrm{d}\phi \ = \int\limits_{\phi}\int\limits_{r} \sqrt{(a^2+1)r^2}\ \mathrm{d}r\,\mathrm{d}\phi \ = \sqrt{a^2+1}\int\limits_{\phi}\int\limits_{r=0}^{B/a} r\ \mathrm{d}r\,\mathrm{d}\phi \tag{336}$$

$$= \sqrt{a^2+1}\int\limits_{\phi} \left[\tfrac{1}{2}r^2\right]_0^{B/a}\ \mathrm{d}\phi \ = \sqrt{a^2+1}\int\limits_{\phi} \tfrac{1}{2}(B/a)^2\ \mathrm{d}\phi \ = \tfrac{1}{2}\frac{B^2\sqrt{a^2+1}}{a^2}\int\limits_{\phi=0}^{2\pi}\mathrm{d}\phi \tag{337}$$

$$= \tfrac{1}{2}B^2\frac{\sqrt{a^2+1}}{a^2}2\pi. \tag{338}$$

□ 37.2 From $x_2 = F(x_0, x_1)$ we obtain:

$$\mathbf{x}_0 = \frac{\partial \mathbf{x}}{\partial x_0} = \left(1, 0, \frac{\partial F}{\partial x_0}\right);\ \ \mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial x_1} = \left(0, 1, \frac{\partial F}{\partial x_1}\right) \tag{339}$$

so

$$g_{00} = 1 + \left(\frac{\partial F}{\partial x_0}\right)^2;\ \ g_{11} = 1 + \left(\frac{\partial F}{\partial x_1}\right)^2;\ \ g_{10} = g_{01} = \frac{\partial F}{\partial x_0}\cdot\frac{\partial F}{\partial x_1} \tag{340}$$

and

$$g = \begin{vmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{vmatrix} \tag{341}$$

$$= \left(1 + \frac{\partial F}{\partial x_0}\right)^2 \cdot \left(1 + \frac{\partial F}{\partial x_1}\right)^2 - \left(\frac{\partial F}{\partial x_0}\cdot\frac{\partial F}{\partial x_1}\right)^2 \tag{342}$$

$$= 1 + \left(\frac{\partial F}{\partial x_1}\right)^2 + \left(\frac{\partial F}{\partial x_0}\right)^2 + \left(\frac{\partial F}{\partial x_0}\right)^2\left(\frac{\partial F}{\partial x_1}\right)^2 - \left(\frac{\partial F}{\partial x_0}\cdot\frac{\partial F}{\partial x_1}\right)^2 \tag{343}$$

$$= 1 + \left(\frac{\partial F}{\partial x_0}\right)^2 + \left(\frac{\partial F}{\partial x_1}\right)^2 \tag{344}$$

so

$$dA = \sqrt{g}\; dx_0\, dx_1 = \sqrt{1 + \left(\frac{\partial F}{\partial x_0}\right)^2 + \left(\frac{\partial F}{\partial x_1}\right)^2}\; dx_0\, dx_1 \tag{345}$$

□ 37.3 Using the representation from Exercise 28.1:

$$g = \begin{vmatrix} g_{\phi\phi} & g_{\phi z} \\ g_{z\phi} & g_{zz} \end{vmatrix} = \begin{vmatrix} r^2(z) & 0 \\ 0 & r'^2(z)+1 \end{vmatrix} = r^2(z)\left(r'^2(z)+1\right) \tag{346}$$

so the element of area is

$$dA = \sqrt{g}\; d\phi\, dz = r(z)\sqrt{r'^2(z)+1} \tag{347}$$

For a sphere of radius R, $r(z)$ is expressed as follows:

$$R^2 = z^2 + r^2 \Rightarrow \quad r^2 = R^2 - z^2 \Rightarrow \quad r = \sqrt{R^2 - z^2} \tag{348}$$

so

$$r'(z) = \tfrac{1}{2}\frac{1}{\sqrt{R^2-z^2}} \cdot -2z = -\frac{z}{\sqrt{R^2-z^2}} \tag{349}$$

and

$$r'^2(z) = \frac{z^2}{R^2-z^2} \Rightarrow \quad r'^2(z)+1 = \frac{R^2}{R^2-z^2} \tag{350}$$

and

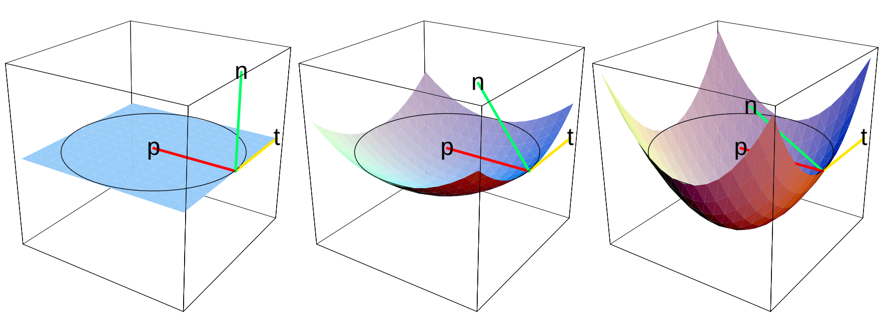
$$g = r^2(z)\left(r'^2(z)+1\right) = \quad (R^2-z^2)\frac{R^2}{R^2-z^2} = \quad R^2 \Rightarrow \quad \sqrt{g} = R \tag{351}$$

Integrating:

$$A = \iint_U dA \;= \int_{\phi=0}^{2\pi}\int_{z=-R}^{+R} R\; dz\, d\phi \;= R\int_{\phi}\int_{z=-R}^{+R} dz\, d\phi \;= R\int_{\phi} [z]_{-R}^{+R}\; d\phi \tag{352}$$

$$= R\int_{\phi=0}^{2\pi} 2R\; d\phi \;= 2R^2\int_{\phi=0}^{2\pi} d\phi \;= 2R^2\,[\phi]_0^{2\pi} \;= 2R^2\cdot 2\pi \;= 4\pi R^2. \tag{353}$$

§38 Second fundamental form.



In $\cos\gamma = \mathbf{p}\cdot\mathbf{n}$, $\mathbf{p}$ is a property of the curve and $\mathbf{n}$ is a property of the plane (see figures). With $\mathbf{p} = \ddot{\mathbf{x}}/\kappa$ this can be written in terms of two different properties of the curve as $\kappa\cos\gamma = \ddot{\mathbf{x}}\cdot\mathbf{n}$.

Defining $b_{\alpha\beta} = \mathbf{x}_{\alpha\beta}\cdot\mathbf{n} = -\mathbf{x}_\alpha\cdot\mathbf{n}_\beta$ as properties of the surface alone, it is derived that

$$\kappa\cos\gamma = \frac{b_{\alpha\beta}\ \mathrm{d}u^\alpha/\mathrm{d}t\ \ \mathrm{d}u^\beta/\mathrm{d}t}{g_{\alpha\beta}\ \mathrm{d}u^\alpha/\mathrm{d}t\ \ \mathrm{d}u^\beta/\mathrm{d}t} = \frac{b_{\alpha\beta}\mathrm{d}u^\alpha\mathrm{d}u^\beta}{g_{\alpha\beta}\mathrm{d}u^\alpha\mathrm{d}u^\beta} \tag{354}$$

Apparently the $\mathrm{d}u^\alpha\mathrm{d}u^\beta$ depend only on the *direction* of the unit tangent vector of the curve, therefore κ depends only on properties of the surface and the directions of the unit tangent and the unit principal normal of the curve.

☐ 38.1 By Exercise 38.8

$$\mathbf{x}_0 = \frac{\partial\mathbf{x}}{\partial x_0} = [1\ \ 0\ \ F_0];\ \ \mathbf{x}_1 = \frac{\partial\mathbf{x}}{\partial x_1} = [0\ \ 1\ \ F_1];\ \ \mathbf{x}_2 = \frac{\partial\mathbf{x}}{\partial x_2} = [0\ \ 0\ \ F_{\alpha\beta}] \tag{355}$$

where $F_\alpha = \partial F/\partial x_\alpha$ and $F_{\alpha\beta} = \partial^2 F/\partial x_\alpha\partial x_\beta$. Then

$$|\mathbf{x}_0\ \ \mathbf{x}_1\ \ \mathbf{x}_{\alpha\beta}| = \begin{vmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ F_0 & F_1 & F_{\alpha\beta}\end{vmatrix} = F_0\begin{vmatrix}0 & 0\\ 1 & 0\end{vmatrix} - F_1\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix} + F_{\alpha\beta}\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} = F_{\alpha\beta} \tag{356}$$

From Exercise 37.2, $g = 1 + (F_0)^2 + (F_1)^2$, so

$$b_{\alpha\beta} = \frac{F_{\alpha\beta}}{\sqrt{1+(F_0)^2+(F_1)^2}}. \tag{357}$$

☐ 38.2 Given an allowable transformation between u^α and $\tilde{u}^\alpha$:

$$\tilde{b}_{\alpha\beta} \overset{(38.4)}{=} -\tilde{\mathbf{x}}_\alpha\cdot\tilde{\mathbf{x}}_\beta = -\frac{\partial\mathbf{x}}{\partial\tilde{u}^\alpha}\cdot\frac{\partial\mathbf{n}}{\partial\tilde{u}^\beta} \tag{358}$$

$$= -\frac{\partial\mathbf{x}}{\partial u^\mu}\frac{\partial u^\mu}{\partial\tilde{u}^\alpha}\cdot\frac{\partial\mathbf{n}}{\partial u^\nu}\frac{\partial u^\nu}{\partial\tilde{u}^\beta} = -\frac{\partial\mathbf{x}}{\partial u^\mu}\cdot\frac{\partial\mathbf{n}}{\partial u^\nu}\frac{\partial u^\mu}{\partial\tilde{u}^\alpha}\frac{\partial u^\nu}{\partial\tilde{u}^\beta} \tag{359}$$

$$= -\mathbf{x}_\mu\cdot\mathbf{n}_\nu\frac{\partial u^\mu}{\partial\tilde{u}^\alpha}\frac{\partial u^\nu}{\partial\tilde{u}^\beta} = b_{\mu\nu}\frac{\partial u^\mu}{\partial\tilde{u}^\alpha}\frac{\partial u^\nu}{\partial\tilde{u}^\beta} \tag{360}$$

which is the transformation of a covariant tensor as in (30.3) or (30.5).

§39 Arbitrary and normal sections of a surface. Meusnier's theorem. Asymptotic lines.

☐ 39.1 A cone of revolution can be represented as

$$\mathbf{x} = \begin{bmatrix}\alpha z\sin\theta\\ \alpha z\cos\theta\\ 0\end{bmatrix};\quad \text{so that } \mathbf{x}_\theta = \begin{bmatrix}-\alpha z\sin\theta\\ \alpha z\cos\theta\\ 0\end{bmatrix},\ \ \mathbf{x}_z = \begin{bmatrix}\alpha\cos\theta\\ \alpha\sin\theta\\ 1\end{bmatrix} \tag{361}$$

So the components of the metric tensor are

$$g_{\theta\theta} = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = +\alpha^2z^2\sin^2\theta + \alpha^2z^2\cos^2\theta + 0 = \alpha^2z^2 \tag{362}$$

$$g_{zz} = \mathbf{x}_z\cdot\mathbf{x}_z = \alpha^2\cos^2\theta + \alpha^2\sin^2\theta + 1 = \alpha^2(\cos^2\theta+\sin^2\theta) + 1 = \alpha^2+1 \tag{363}$$

$$g_{z\theta} = g_{\theta z} = \begin{bmatrix}-\alpha z\sin\theta\\ \alpha z\cos\theta\\ 0\end{bmatrix}\cdot\begin{bmatrix}\alpha\cos\theta\\ \alpha\sin\theta\\ 1\end{bmatrix} = -\alpha^2z\sin\theta\cos\theta + \alpha^2z\sin\theta\cos\theta + 0 = 0 \tag{364}$$

Since

$$\mathbf{x}_\theta \times \mathbf{x}_z = \begin{bmatrix} -\alpha z \sin\theta \\ \alpha z \cos\theta \\ 0 \end{bmatrix} \times \begin{bmatrix} \alpha\cos\theta \\ \alpha\sin\theta \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha z\cos\theta \\ +\alpha z \sin\theta \\ -\alpha z\sin\theta \cdot \alpha\sin\theta - \alpha\cos\theta\,\alpha z\cos\theta \end{bmatrix} \tag{365}$$

$$= \begin{bmatrix} \alpha z\cos\theta \\ \alpha z\sin\theta \\ -\alpha^2 z\sin^2\theta - \alpha^2 z\cos^2\theta \end{bmatrix} = \begin{bmatrix} \alpha z\cos\theta \\ \alpha z\sin\theta \\ -\alpha^2 z \end{bmatrix} \tag{366}$$

and

$$g = \begin{vmatrix} \alpha^2 z^2 & 0 \\ 0 & \alpha^2+1 \end{vmatrix} = \alpha^2 z^2(\alpha^2+1) \quad \Rightarrow \sqrt{g} = \sqrt{\alpha^2 z^2(\alpha^2+1)} = \alpha z\sqrt{\alpha^2+1} \tag{367}$$

So by Equation 34.1:

$$\mathbf{n} = \frac{1}{\alpha z\sqrt{\alpha^2+1}} \begin{bmatrix} \alpha z\cos\theta \\ \alpha z\sin\theta \\ -\alpha^2 z \end{bmatrix} = \frac{1}{\sqrt{\alpha^2+1}} \begin{bmatrix} \cos\theta \\ \sin\theta \\ -\alpha \end{bmatrix} \tag{368}$$

is independent of z. Since the generating lies of the cone are those with constant θ, all the points on a generating line have tangent planes with identical mormal vector-- hence their normal planes coincide.

□ 39.3 A cylinder of revolution can be represented as

$$\mathbf{x} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}; \quad \text{so that} \tag{369}$$

$$\mathbf{x}_\theta = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix}, \quad \mathbf{x}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad \text{and} \tag{370}$$

$$\mathbf{x}_{\theta\theta} = \begin{bmatrix} -r\cos\theta \\ -r\sin\theta \\ 0 \end{bmatrix}, \quad \mathbf{x}_{zz} = 0, \quad \mathbf{x}_{\theta z} = \mathbf{x}_{z\theta} = 0 \tag{371}$$

so by Equation 38.2 $b_{zz} = 0, b_{\theta z} = b_{z\theta} = 0$. By (39.4) the differential equation of any asymptotic curves are $b_{\theta\theta}\,\mathrm{d}\theta\mathrm{d}\theta = 0$. Since by 38.8

$$b_{\theta\theta} = \frac{1}{\sqrt{g}} \begin{vmatrix} \mathbf{x}_\theta & \mathbf{x}_z & \mathbf{x}_{\theta\theta} \end{vmatrix} \tag{372}$$

$$= \frac{1}{\sqrt{g}} = \begin{vmatrix} -r\sin\theta & 0 & -r\cos\theta \\ r\cos\theta & 0 & -r\sin\theta \\ 0 & 1 & 0 \end{vmatrix} = \frac{1}{\sqrt{g}} \cdot -1 \begin{vmatrix} -r\sin\theta & -r\cos\theta \\ r\cos\theta & -r\sin\theta \end{vmatrix} \tag{373}$$

$$= -\frac{1}{\sqrt{g}}\left(+r^2\sin^2\theta + r^2\cos^2\theta\right) = -\frac{1}{\sqrt{g}} \cdot r^2 = -\frac{r^2}{\sqrt{g}} \neq 0 \tag{374}$$

the asymptotic curves must then have $\mathrm{d}\theta = 0$, so the lines with constant θ are the asymptotic curves.

§40 Elliptic, parabolic, and hyperbolic points of a surface.

□ 40.1 Beginning again with a representation of a surface of revolution:

$$\mathbf{x}(\theta, r) = [r\cos\theta \quad r\sin\theta \quad z(r)] \tag{375}$$

we derive first and second derivatives:

$$\mathbf{x}_\theta = [-r\sin\theta \quad r\cos\theta \quad 0]\,; \quad \mathbf{x}_r = [\cos\theta \quad \sin\theta \quad z'(r)] \tag{376}$$

$$\mathbf{x}_{\theta\theta} = [-r\cos\theta \quad -r\sin\theta \quad 0] \quad \mathbf{x}_{rr} = [0 \quad 0 \quad z''(r)]\,; \quad \mathbf{x}_{r\theta} = \mathbf{x}_{\theta r} = [-\sin\theta \quad \cos\theta \quad 0] \tag{377}$$

so then the components of the first fundamental form are:

$$g_{\theta\theta} = +r^2\sin^2\theta + r^2\cos^2\theta + 0 = r^2 \tag{378}$$

$$g_{rr} = \cos^2\theta + \sin^2\theta + z'^2(r) = z'^2(r) + 1 \tag{379}$$

$$g_{r\theta} = g_{\theta r} = -r\sin\theta\cos\theta + r\sin\theta\cos\theta + 0 = 0 \tag{380}$$

and

$$g = |_{\alpha\beta}\ g_{\alpha\beta}| = \begin{vmatrix} r^2 & 0 \\ 0 & z'^2(r)+1 \end{vmatrix} = r^2\left(z'^2(r)+1\right) \tag{381}$$

Then the components of the second fundamental form are

$$b_{\theta\theta} = \frac{1}{\sqrt{g}}\begin{vmatrix} -r\sin\theta & \cos\theta & -r\cos\theta \\ r\cos\theta & \sin\theta & -r\sin\theta \\ 0 & z'(r) & 0 \end{vmatrix} = \frac{z'(r)}{\sqrt{g}} = \begin{vmatrix} -r\sin\theta & -r\cos\theta \\ r\cos\theta & -r\sin\theta \end{vmatrix} \tag{382}$$

$$= -\frac{z'(r)}{\sqrt{g}}\left(+r^2\sin^2\theta + r^2\cos^2\theta\right) = -\frac{r^2 z'(r)}{\sqrt{g}} = -\frac{r^2 z'(r)}{r\sqrt{z'^2(r)+1}} = -\frac{rz'(r)}{\sqrt{z'^2(r)+1}} \tag{383}$$

$$b_{rr} = \frac{1}{\sqrt{g}}\begin{vmatrix} -r\sin\theta & \cos\theta & 0 \\ r\cos\theta & \sin\theta & 0 \\ 0 & z'(r) & z''(r) \end{vmatrix} = \frac{z''(r)}{\sqrt{g}}\begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} \tag{384}$$

$$= \frac{z''(r)}{\sqrt{g}}\left(-r\sin^2\theta - r\cos^2\theta\right) = -\frac{rz''(r)}{\sqrt{g}} = -\frac{rz''(r)}{r\sqrt{z'^2(r)+1}} = -\frac{z''(r)}{\sqrt{z'^2(r)+1}} \tag{385}$$

$$b_{\theta r} = b_{r\theta} = \frac{1}{\sqrt{g}}\begin{vmatrix} -r\sin\theta & \cos\theta & -\sin\theta \\ r\cos\theta & \sin\theta & \cos\theta \\ 0 & z'(r) & 0 \end{vmatrix} = -\frac{z'(r)}{\sqrt{g}}\begin{vmatrix} -r\sin\theta & -\sin\theta \\ r\cos\theta & \cos\theta \end{vmatrix} \tag{386}$$

$$= -\frac{z'(r)}{\sqrt{g}}\left(-r\sin\theta\cos\theta + r\sin\theta\cos\theta\right) = 0 \tag{387}$$

The asymptotic curves therefore are given from (39.4) by

$$b_{\alpha\beta}\ \mathrm{d}u^\alpha \mathrm{d}u^\beta = b_{\theta\theta}\ \mathrm{d}\theta\mathrm{d}\theta + b_{rr}\ \mathrm{d}r\mathrm{d}r \tag{388}$$

$$= \frac{-1}{\sqrt{z'^2(r)+1}}\left(rz'(r)\ \mathrm{d}\theta\mathrm{d}\theta + z''(r)\ \mathrm{d}r\mathrm{d}r\right) = 0 \tag{389}$$

$$\Rightarrow rz'(r)\ \mathrm{d}\theta\mathrm{d}\theta + z''(r)\ \mathrm{d}r\mathrm{d}r = 0 \tag{390}$$

Now

$$b = |_{\alpha\beta} b_{\alpha\beta}| \tag{391}$$

$$= \begin{vmatrix} -\frac{r^2 z'(r)}{\sqrt{g}} & 0 \\ 0 & -\frac{rz''(r)}{\sqrt{g}} \end{vmatrix} = \frac{1}{g}\begin{vmatrix} r^2 z'(r) & 0 \\ 0 & rz''(r) \end{vmatrix} = \frac{r^3 z'(r) z''(r)}{r^2\left(z'^2(r)+1\right)} = r\frac{z'(r)z''(r)}{z'^2(r)+1} \tag{392}$$

Therefore, if $z'(r) = 0$ then $b = 0$ and points are parabolic. If $z'(r)z''(r) > 0$ points are elliptic; if $z'(r)z''(r) < 0$, they are parabolic.

§41 Principal curvature. Lines of curvature. Gaussian and mean curvature.

□ 41.1 For a general surface of revolution, Exercise 28.1 gives a representation for which $g_{z\phi} = g_{\phi z} = 0$. Also:

$$b_{z\phi} = b_{\phi z} = \frac{1}{\sqrt{g}} \begin{vmatrix} \mathbf{x}_z & \mathbf{x}_\phi & \mathbf{x}_{z\phi} \end{vmatrix} \tag{393}$$

where, building still on Exercise 28.1:

$$\mathbf{x}_{z\phi} = \frac{\partial \mathbf{x}_z}{\partial \phi} = \frac{\partial}{\partial \phi} \begin{bmatrix} r'(z)\cos\phi & r'(z)\sin\phi & 1 \end{bmatrix} = \begin{bmatrix} -r'(z)\sin\phi & r'(z)\cos\phi & 0 \end{bmatrix} \tag{394}$$

so

$$\begin{vmatrix} \mathbf{x}_z & \mathbf{x}_\phi & \mathbf{x}_{z\phi} \end{vmatrix} = \begin{vmatrix} r'(z)\cos\phi & -r(z)\sin\phi & -r'(z)\sin\phi \\ r'(z)\sin\phi & r(z)\cos\phi & r'(z)\cos\phi \\ 1 & 0 & 0 \end{vmatrix} = +1 \begin{vmatrix} -r(z)\sin\phi & -r'(z)\sin\phi \\ r(z)\cos\phi & r'(z)\cos\phi \end{vmatrix} \tag{395}$$

$$= -r(z)r'(z)\sin\phi\cos\phi + r(z)r'(z)\sin\phi\cos\phi = 0 \tag{396}$$

Therefore by Theorem 41.3, the coordinate curves under this representation are also lines of curvature. For the special case of surfaces which are (locally) cylindrical, the previous representation doesn't apply. In that case we can use the representation of Exercise 39.3, where $g_{\theta z} = g_{z\theta} = 0$ because the coordinate curves are orthogonal (Theorem 35.1) and $b_{\theta z} = b_{z\theta} = 0$, so that the coordinate curves are lines of curvature in this case also.

□ 41.2 From Exercise 27.4 we have the components of the first fundamental form of a sphere, so

$$g = \begin{vmatrix} g_{\theta\theta} & g_{\phi\theta} \\ g_{\theta\phi} & g_{\phi\phi} \end{vmatrix} = \begin{vmatrix} r^2 & 0 \\ 0 & r^2\cos^2\theta \end{vmatrix} = r^4\cos^2\theta \tag{397}$$

The normal curvature κ_{N} at any point is simply $1/r$, so the total ('Gaussian') curvature is $\kappa = 1/r^2$. Thus

$$K = \frac{b}{g} \quad \Rightarrow b = gK = \frac{r^4\cos^2\theta}{r^2} = r^2\cos^2\theta \tag{398}$$

§45 Formulae of Weingarten and Gauss.

△ The objective is to find a representation of the partial derivatives of $\mathbf{x}_\alpha$ and $\mathbf{n}$ with respect to u^α in terms of themselves, along the lines of the formulae of Frenet for curves.

Beginning with

$$\mathbf{n}\cdot\mathbf{n} = 1 \quad \Rightarrow \frac{\mathrm{d}}{\mathrm{d}u^\alpha}(\mathrm{n}\cdot\mathrm{n}) = \mathrm{n}_\alpha\cdot\mathrm{n} + \mathrm{n}\cdot\mathrm{n}_\alpha = 2(\mathrm{n}_\alpha\cdot\mathrm{n}) = 0 \quad \Rightarrow \mathrm{n}_\alpha\cdot\mathrm{n} = 0 \tag{399}$$

Since $\mathbf{n}_\alpha \perp \mathbf{n}$, they are in the tangent plane. So

$$\exists c_\alpha^\gamma : \quad \mathbf{n}_\alpha = c_\alpha^\gamma\,\mathbf{x}_\gamma \tag{400}$$

then

$$\forall \mathbf{x}_\sigma : \quad \mathbf{n}_\alpha\cdot\mathbf{x}_\sigma = c_\alpha^\gamma\,\mathbf{x}_\gamma\cdot\mathbf{x}_\sigma = c_\alpha^\gamma g_{\gamma\sigma} \tag{401}$$

Since $b_{\sigma\alpha} = -\mathbf{x}_\sigma \cdot \mathbf{n}_\alpha$ (38.4):

$$-b_{\sigma\alpha} = c_\alpha^\gamma g_{\gamma\sigma} \tag{402}$$

and then

$$-b_{\sigma\alpha} \cdot g^{\sigma\tau} = c_\alpha^\gamma g_{\gamma\sigma} g^{\sigma\tau} = c_\alpha^\gamma \delta_\gamma^\tau = c_\alpha^\tau \quad \Rightarrow -b_\alpha^\tau = c_\alpha^\tau \quad \Rightarrow b_\alpha^\tau = -c_\alpha^\tau \tag{403}$$

so that

$$\mathbf{n}_\alpha = -b_\alpha^\beta \, \mathbf{x}_\beta \tag{404}$$

Since in general $\mathbf{x}_\alpha \cdot \mathbf{x}_\beta = 1$ a derivative $\mathbf{x}_{\alpha\beta}$ will not lie in a plane orthogonal to $\mathbf{x}_\alpha$; so

$$\exists \Gamma_{\alpha\beta}^\gamma, a_{\alpha\beta} : \quad \mathbf{x}_{\alpha\beta} = \Gamma_{\alpha\beta}^\gamma \mathbf{x}_\gamma + a_{\alpha\beta} \, \mathbf{n} \tag{405}$$

To determine the $a_{\alpha\beta}$ we take

$$\mathbf{x}_{\alpha\beta} \cdot \mathbf{n} = \Gamma_{\alpha\beta}^\gamma \, \mathbf{x}_\gamma \cdot \mathbf{n} + a_{\alpha\beta} \cdot \mathbf{n} \cdot \mathbf{n} \tag{406}$$

Since by definition $b_{\alpha\beta} = \mathbf{x}_{\alpha\beta} \cdot \mathbf{n}$ (38.2), $\mathbf{x}_\gamma \cdot \mathbf{n} = 0$, and $\mathbf{n} \cdot \mathbf{n} = 1$:

$$b_{\alpha\beta} = 0 + a_{\alpha\beta} \quad \Rightarrow a_{\alpha\beta} = b_{\alpha\beta} \tag{407}$$

To determine the $\Gamma_{\alpha\beta}^\gamma$ we take

$$\mathbf{x}_{\alpha\beta} \cdot \mathbf{x}^\kappa = \Gamma_{\alpha\beta}^\gamma \, \mathbf{x}_\gamma \cdot \mathbf{x}^\kappa + a_{\alpha\beta} \, \mathbf{n} \cdot \mathbf{x}^\kappa \tag{408}$$

Since $\mathbf{x}_\gamma \cdot \mathbf{x}^\kappa = \delta_\gamma^\kappa$ and again $\mathbf{n} \perp \mathbf{x}^\kappa = g^{\rho\kappa} \mathbf{x}_\rho$:

$$\mathbf{x}_{\alpha\beta} \cdot \mathbf{x}^\kappa = \Gamma_{\alpha\beta}^\gamma \delta_\gamma^\kappa + 0 \quad \Rightarrow \Gamma_{\alpha\beta}^\kappa = \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}^\kappa \tag{409}$$

so

$$\mathbf{x}_{\alpha\beta} = \Gamma_{\alpha\beta}^\gamma \mathbf{x}_\gamma + b_{\alpha\beta} \, \mathbf{n}; \quad \Gamma_{\alpha\beta}^\gamma = \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}^\gamma \tag{410}$$

□ 45.1 If $b_{\alpha\beta} = 0$ then $\mathbf{x}_{\alpha\beta} = \Gamma_{\alpha\beta}^\gamma \, \mathbf{x}_\gamma$, so over the entire surface the $\mathbf{x}_\alpha$ remain in the plane spanned by $\mathbf{x}_\alpha$; ergo the surface is a plane.